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N-DIMENSION CENTRAL AFFINE CURVE FLOWSCHUU-LIAN TERNG[†] AND ZHIWEI WU*

1. INTRODUCTION

The group $SL(n, \mathbb{R})$ acts on $\mathbb{R}^n \setminus \{0\}$ transitively by $g \cdot y = gy$ for $g \in SL(n, \mathbb{R})$ and $y \in \mathbb{R}^n$. Given a curve γ in $\mathbb{R}^n \setminus \{0\}$, if $\det(\gamma, \gamma_s, \dots, \gamma_s^{(n-1)})$ is positive then there is an orientation preserving parameter x unique up to translation, $\frac{dx}{ds} = \det(\gamma, \gamma_s, \dots, \gamma_s^{(n-1)})^{\frac{2}{n(n-1)}}$, such that

$$\det(\gamma, \gamma_x, \dots, \gamma_x^{(n-1)}) = 1, \quad (1.1)$$

where $\gamma_x^{(i)} = \frac{d^i \gamma}{dx^i}$. Take x derivative of (1.1) to get

$$\det(\gamma, \gamma_x, \dots, \gamma_x^{(n-2)}, \gamma_x^{(n)}) = 0.$$

Hence we have

$$\gamma_x^{(n)} = u_1 \gamma + u_2 \gamma_x + \dots + u_{n-1} \gamma_x^{(n-2)},$$

where $u_i = \det(\gamma, \gamma_x, \dots, \gamma_x^{(i-2)}, \gamma_x^{(n)}, \gamma_x^{(i)}, \dots, \gamma_x^{(n-1)})$. This parameter x is called the *central affine arc-length parameter*,

$$g = (\gamma, \dots, \gamma_x^{(n-1)})$$

the *central affine moving frame*, and u_i the *i -th central affine curvature* of γ for $1 \leq i \leq n-1$ (cf. [3], [8]). Note that

$$g_x = g(b + u),$$

where $b = \sum_{i=1}^{n-1} e_{i+1,i}$ and $u = \sum_{i=1}^{n-1} u_i e_{in}$. We also call u the *central affine curvature* along γ .

Let $I = S^1$ or \mathbb{R} , and

$$\mathcal{M}_n(I) = \{\gamma : I \rightarrow \mathbb{R}^n \setminus \{0\} \mid \det(\gamma, \gamma_x, \dots, \gamma_x^{(n-1)}) = 1\},$$

$$V_n = \oplus_{i=1}^{n-1} \mathbb{R} e_{in} \subset sl(n, \mathbb{R}).$$

Let $\Psi : \mathcal{M}_n(I) \rightarrow C^\infty(I, V_n)$ be the *central affine curvature map* defined by

$$\Psi(\gamma) = u = \sum_{i=1}^{n-1} u_i e_{in},$$

where u_1, \dots, u_{n-1} are the central affine curvatures along γ .

It follows from the Existence and Uniqueness for Ordinary Differential Equations that Ψ induces a bijection from the orbit space $\frac{\mathcal{M}_n(\mathbb{R})}{SL(n, \mathbb{R})}$ to $C^\infty(\mathbb{R}, V_n)$

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and u_1, \dots, u_{n-1} form a complete set of local differential invariants for curves in $\mathbb{R}^n \setminus \{0\}$ under the group $SL(n, \mathbb{R})$. For example, curves with zero central affine curvatures in $\mathbb{R}^n \setminus \{0\}$ are of the form

$$\gamma(x) = c \left(1, x, \frac{x^2}{2}, \dots, \frac{x^{n-1}}{(n-1)!} \right)^t$$

for some $c \in SL(n, \mathbb{R})$.

Let $y \in C^\infty(\mathbb{R}, \mathbb{R}^n)$. We say η is a *differential polynomial of order k in y* if η is a polynomial in $y, y_x, \dots, y_x^{(k)}$.

A vector field $X : \mathcal{M}_n(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}, \mathbb{R}^n)$ is tangent to $\mathcal{M}_n(\mathbb{R})$ if and only if

$$\sum_{i=0}^{n-1} \det(\gamma, \dots, \gamma_x^{(i-1)}, (X(\gamma))_x^{(i)}, \dots, \gamma_x^{(n-1)}) = 0. \quad (1.2)$$

We show in section 2 that there exists a differential polynomial ϕ_n in $u, \xi_1, \dots, \xi_{n-1}$ such that the vector field $\sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ if and only if

$$\xi_0 = \phi_n(u, \xi_1, \dots, \xi_{n-1}),$$

where u is the central affine curvature along γ . So $T\mathcal{M}_n(\mathbb{R})_\gamma$ is identified as $C^\infty(\mathbb{R}, \mathbb{R}^{n-1})$.

A *central affine curve flow* is an evolution equation on $\mathcal{M}_n(\mathbb{R})$ of the form

$$\gamma_t = X(\gamma) = \xi_0(u)\gamma + \xi_1(u)\gamma_x + \dots + \xi_{n-1}(u)\gamma_x^{(n-1)}, \quad (1.3)$$

where $X(\gamma)$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ and ξ_0, \dots, ξ_{n-1} are differential polynomials in the central affine curvature $u(\cdot, t)$ of $\gamma(\cdot, t)$. So $\xi_0(u) = \phi_n(u, \xi_1(u), \dots, \xi_{n-1}(u))$.

It is easy to see that a central affine curve flow is invariant under the action of $SL(n, \mathbb{R})$ on $\mathbb{R}^n \setminus \{0\}$ and translations in the (x, t) -plane. In other words, if $\gamma(x, t)$ is a solution of (1.3), then so is $\tilde{\gamma}(x, t) = c\gamma(x + r_1, t + r_2)$, where $c \in SL(n, \mathbb{R})$ and $r_1, r_2 \in \mathbb{R}$ are constants.

Let $n \geq 3$. Equation (1.2) implies that $X(\gamma) = y_1(\gamma)\gamma + \gamma_{xx}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ if and only if $y_1(\gamma) = -\frac{2}{n}u_{n-1}$. So

$$\gamma_t = -\frac{2}{n}u_{n-1}\gamma + \gamma_{xx}, \quad (1.4)$$

is one of the simplest central affine curve flows on $\mathcal{M}_n(\mathbb{R})$ for $n \geq 3$, where $u_{n-1}(\cdot, t)$ is the $(n-1)$ -th central affine curvature for $\gamma(\cdot, t)$. This curve flow turns out to be integrable. In fact, if γ is a solution of the central affine curve flow (1.4) then its central affine curvature $u(\cdot, t)$ is a solution of the second flow of the Drinfeld-Sokolov $A_{n-1}^{(1)}$ -KdV hierarchy constructed in [5]. The $A_{n-1}^{(1)}$ -KdV hierarchy is the same as the Gelfand-Dickey (GD_n) hierarchy on the space of n -th order differential operators on the line. So the central affine curve flow (1.4) can be viewed as a simple and natural geometric interpretation of the second GD_n flow.

Next we state some of our main results:

- (a) We construct a sequence of commuting higher order central affine curve flows on $\mathcal{M}_n(\mathbb{R})$ such that the second flow is (1.4). For example, the third central affine curve flow on $\mathcal{M}_n(\mathbb{R})$ ($n \neq 3$) is

$$\gamma_t = \left(-\frac{3}{n}u_{n-2} + \frac{3(n-3)}{2n}(u_{n-1})_x \right) \gamma - \frac{3}{n}u_{n-1}\gamma_x + \gamma_{xxx}. \quad (1.5)$$

Note that when $n = 2$, we have $\gamma_{xx} = u_1\gamma$. Then (1.5) becomes

$$\gamma_t = \frac{1}{4}(u_1)_x\gamma - \frac{1}{2}u_1\gamma_x. \quad (1.6)$$

It was proved in [8] that if γ is a solution of (1.6) on $\mathcal{M}_2(\mathbb{R})$ then u_1 is a solution of the KdV equation.

- (b) We prove that the central affine curvature map Ψ gives a one to one correspondence between solutions of the j -th central affine curve flow modulo the action of $SL(n, \mathbb{R})$ and solutions of the j -th $A_{n-1}^{(1)}$ -KdV flow.
- (c) We use the solution of the Cauchy problem of the $A_{n-1}^{(1)}$ -KdV hierarchy to solve the Cauchy problem for the central affine curve flow hierarchy with periodic initial data and also with initial data having rapidly decaying central affine curvatures.
- (d) We construct Bäcklund transformations, a permutability formula, and infinitely many families of explicit solutions of the central affine curve flows.

We pull back the known sequence of Poisson structures for the $A_{n-1}^{(1)}$ -KdV hierarchy via the central affine curvature map Ψ to $\mathcal{M}_n(S^1)$ for the central affine curve flow hierarchy. We also prove the following results in this paper:

- (1) The central affine curve flows (1.4) and (1.5) are the Hamiltonian equations for the functionals

$$F_2(\gamma) = \oint u_{n-2}(x)dx,$$

$$F_3(\gamma) = \oint u_{n-3} + \frac{n-3}{2n}u_{n-1}^2 dx,$$

with respect to the second Poisson structure respectively, where u_i is the i -th central affine curvature along γ .

- (2) The second and third Poisson structures arise naturally from coadjoint orbits.
- (3) We identify the kernels of these two Poisson operators.
- (4) Since the \mathbb{R}^{n-1} -action on $\mathcal{M}_n(S^1)$ generated by the first $(n-1)$ central affine curve flows commutes with the $SL(n, \mathbb{R})$ -action, the direct product $SL(n, \mathbb{R}) \times \mathbb{R}^{n-1}$ acts on $\mathcal{M}_n(S^1)$. We show that the second and third Poisson structures on $\mathcal{M}_n(S^1)$ induces weak symplectic forms on the orbits space $\frac{\mathcal{M}_n(S^1)}{SL(n, \mathbb{R})}$ and $\frac{\mathcal{M}_n(S^1)}{SL(n, \mathbb{R}) \times \mathbb{R}^{n-1}}$ respectively.

When $n = 3$, result (a) and one Poisson structure were obtained in [3] for (1.4). When $n = 2$, result (4) was proved in [6].

This paper is organized as follows: In section 2, we give a brief review of the $A_{n-1}^{(1)}$ -KdV hierarchy and prove some properties of its Lax pair. We prove results (a)-(c) in section 3 and (d) in section 4. We review the bi-Hamiltonian structure of the $A_{n-1}^{(1)}$ -KdV hierarchy and compute the kernels of the bi-Hamiltonian structures in section 5. We write down the formula for the bi-Hamiltonian structures of the curve flow (1.4) and prove results concerning Hamiltonian structures in the last section.

2. THE $A_{n-1}^{(1)}$ -KdV HIERARCHY

Drinfeld and Sokolov constructed a hierarchy of KdV type for each affine Kac-Moody algebra. In this section, we

- (i) give a brief review of the construction of the $A_{n-1}^{(1)}$ -KdV hierarchy (cf. [5]),
- (ii) develop some properties of its Lax pair that are needed for the study of central affine curve flows and the bi-Hamiltonian structure,
- (iii) identify $T\mathcal{M}(\mathbb{R})_\gamma$ as $C^\infty(\mathbb{R}, \mathbb{R}^{n-1})$.

We first set up some notations. Let

$$\begin{aligned}\mathcal{B}_n^+ &= \{y = (y_{ij}) \in sl(n, \mathbb{R}) \mid y_{ij} = 0, i > j\}, \\ \mathcal{N}_n^+ &= \{y = (y_{ij}) \in sl(n, \mathbb{R}) \mid y_{ij} = 0, i \geq j\}, \\ \mathcal{T}_n &= \{y \in gl(n, \mathbb{R}) \mid y_{ij} = 0, i \neq j\},\end{aligned}$$

denote the subalgebras of upper triangular, strictly upper triangular matrices in $sl(n, \mathbb{R})$ and diagonal matrices in $gl(n, \mathbb{R})$ respectively, and N_n^+ the corresponding Lie subgroup of \mathcal{N}_n^+ .

Let

$$\mathcal{L}(sl(n, \mathbb{R})) = \left\{ \xi(\lambda) = \sum_{j \leq n_0} \xi_j \lambda^j \mid \xi_j \in sl(n, \mathbb{R}), n_0 \text{ integer} \right\}.$$

For $\xi \in \mathcal{L}(sl(n, \mathbb{R}))$, we use the following notation:

$$\xi_+ = \sum_{j \geq 0} \xi_j \lambda^j, \quad \xi_- = \sum_{j < 0} \xi_j \lambda^j.$$

Let

$$J = e_{1,n} \lambda + b, \quad b = \sum_{i=1}^{n-1} e_{i+1,i}.$$

Given $u \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, a direct computation (cf. [11]) implies that there exists a unique $Y(u, \lambda) \in \mathcal{L}(sl(n, \mathbb{C}))$ satisfying

$$\begin{cases} [\partial_x + J + u, Y(u, \lambda)] = 0, \\ Y(u, \lambda)^n = \lambda I_n. \end{cases} \quad (2.1)$$

Moreover, the coefficients of the power series expansion of $Y(u, \lambda)$ in λ are differential polynomials of u .

Given $j \not\equiv 0 \pmod{n}$, let $Y(u, \lambda)$ denote the solution of (2.1), and write

$$(Y(u, \lambda))^j = \sum_{-\infty}^{\lfloor \frac{j}{n} \rfloor + 1} Y_{j,i}(u) \lambda^i. \quad (2.2)$$

It was known (cf. [5], [11]) that if $u \in C^\infty(\mathbb{R}, V_n)$ then there is a unique $\zeta_j(u) \in \mathcal{N}_n^+$ such that

$$[\partial_x + b + u, Y_{j,0}(u) - \zeta_j(u)] \in V_n = \oplus_{i=1}^{n-1} \mathbb{R} e_{in} \quad (2.3)$$

and entries of $\zeta_j(u)$ are differential polynomials in u . Set

$$Z_j(u, \lambda) = (Y(u, \lambda)^j)_+ - \zeta_j(u) = \sum_{0 \leq i \leq \lfloor \frac{j}{n} \rfloor + 1} Z_{j,i}(u) \lambda^i. \quad (2.4)$$

The j -th $A_{n-1}^{(1)}$ -KdV hierarchy ($j \geq 0$ and $j \not\equiv 0 \pmod{n}$) constructed by Drinfeld-Sokolov in [5] is the following flow on $C^\infty(\mathbb{R}, V_n)$:

$$u_t = [\partial_x + b + u, Z_{j,0}(u)]. \quad (2.5)$$

Proposition 2.1. ([5]) *Given $u \in C^\infty(\mathbb{R}^2, V_n)$, then the following statements are equivalent*

- (i) u is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow, (2.5),
- (ii) for all parameter $\lambda \in \mathbb{C}$, u satisfies

$$[\partial_x + J + u, \partial_{t_j} + Z_j(u, \lambda)] = 0, \quad (2.6)$$

(i.e. (2.6) is the Lax pair of (2.5)),

- (iii) $[\partial_x + b + u, \partial_t + Z_{j,0}(u)] = 0$, which is the Lax pair (2.6) with parameter $\lambda = 0$,

- (iv) the following system is solvable for $E(x, t, \lambda) \in GL(n, \mathbb{C})$:

$$\begin{cases} E^{-1} E_x = J + u, \\ E^{-1} E_t = Z_j(u, \lambda), \\ \overline{E(x, t, \bar{\lambda})} = E(x, t, \lambda). \end{cases} \quad (2.7)$$

We call a solution E of (2.7) a *frame* of the solution u of the j -th $A_{n-1}^{(1)}$ -KdV flow (2.5). Note that the third condition of (2.7) implies that $E(x, t, \lambda) \in GL(n, \mathbb{R})$ for $\lambda \in \mathbb{R}$.

It follows from $d(\ln \det(E)) = \text{tr}(E^{-1} dE)$ and $\text{tr}(J + u) = \text{tr}(Z_j(u, \lambda)) = 0$ that we have

Corollary 2.2. *If $E(x, t, \lambda)$ is a frame of a solution u of the j -th $A_{n-1}^{(1)}$ -KdV flow, then $\det(E(x, t, \lambda))$ is independent of x, t .*

Next we derive some properties of $Z_{j,0}(u)$. Note that $C = Z_{j,0}(u)$ satisfies

$$[\partial_x + b + u, C] \in V_n. \quad (2.8)$$

The following Theorem shows that if C satisfies (2.8) then C is determined by $\{C_{i1} \mid 2 \leq i \leq n\}$ or by $\{C_{ni} \mid 1 \leq i \leq n-1\}$.

Theorem 2.3. *Let $u \in C^\infty(\mathbb{R}, V_n)$, and C_j the j -th column of $C = (C_{ij}) \in C^\infty(\mathbb{R}, sl(n, \mathbb{R}))$. If C satisfies (2.8), then*

- (i) $C_{j+1} = (C_j)_x + (b+u)C_j$ for $1 \leq j \leq n-1$,
- (ii) *there exists a differential polynomial $\phi_n(u, C_{21}, \dots, C_{n1})$ such that*

$$C_{11} = \phi_n(u, C_{21}, \dots, C_{n1}),$$

- (iii) *entries of C are differential polynomials in u, C_{21}, \dots, C_{n1} ,*
- (iv) *for $1 \leq i \leq n-1$, we have*

$$C_{ni} = C_{n-i+1,1} + (i-1)(C_{n-i+2,1})_x + \phi_i(C_{n1}, \dots, C_{n-i+3,1}), \quad (2.9)$$

for some linear differential operators ϕ_i with differential polynomials in u as coefficients,

- (v) C_{ij} 's are differential polynomials in $u, C_{n1}, \dots, C_{n,n-1}$.

Proof. (i) follows from $C_x + [b+u, C] \in V_n$ and a direct computation.

It follows from (i) and induction on i that there exist differential polynomials $\psi_i(u, C_{21}, \dots, C_{n1})$ such that

$$C_{i+1,i+1} = C_{ii} + \psi_i(u, C_{21}, \dots, C_{n1})$$

for $1 \leq i \leq n-1$. Since $\text{tr}(C) = 0$, we obtain (ii). Statements (iii)-(v) can be proved using (i) and induction. \square

Corollary 2.4. *Let $\gamma \in \mathcal{M}_n(\mathbb{R})$, $u = \sum_{i=1}^{n-1} u_i e_{in}$ the central affine curvature along γ , and $C = (C_{ij}) : \mathbb{R} \rightarrow sl(n, \mathbb{R})$ satisfying (2.8). Then $\xi(\gamma) = \sum_{i=1}^n C_{i1} \gamma_x^{(i-1)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ .*

Proof. Let $g = (\gamma, \dots, \gamma_x^{(n-1)})$ be the central affine moving frame along γ , C_i the i -th column of C , and $\eta_i = gC_i$. Then $g^{-1}g_x = b+u$ and $\xi(\gamma) = gC_1 = \eta_1$. Let $\rho = [\partial_x + b+u, C] = C_x + [b+u, C]$. Then $\rho \in V_n$, and

$$(gC)_x = g_x C + gC_x = gC(b+u) + g\rho. \quad (2.10)$$

Since the first $(n-1)$ columns of ρ are zero, (2.10) implies that $\eta_{i+1} = (\eta_i)_x$ for $1 \leq i \leq n-1$. So we have

$$(\xi(\gamma))_x^{(i-1)} = (\eta_1)_x^{(i-1)} = \eta_i = gC_i.$$

Hence $\det(\gamma, \dots, \gamma_x^{(i-2)}, (\xi(\gamma))_x^{(i-1)}, \gamma_x^{(i)}, \dots, \gamma_x^{(n-1)}) = C_{ii}$. Since C is in $sl(n, \mathbb{R})$, we have $\sum_{i=1}^n C_{ii} = 0$. So $\xi(\gamma)$ satisfies (1.2), i.e.,

$$\sum_{i=1}^n \det(\gamma, \dots, \gamma_x^{(i-2)}, (\xi(\gamma))_x^{(i-1)}, \gamma_x^{(i)}, \dots, \gamma_x^{(n-1)}) = 0.$$

Hence $\xi(\gamma)$ is tangent to $\mathcal{M}(\mathbb{R})$ at γ . \square

Henceforth we set

$$e_1 = (1, 0, \dots, 0)^t \in \mathbb{R}^n.$$

Since $Z_{j,0}(u)$ satisfies (2.8), we have the following.

Corollary 2.5. *Given $\gamma \in \mathcal{M}_n(\mathbb{R})$, let g and u be the central affine moving frame and curvature of γ respectively. Then $\xi(\gamma) = gZ_{j,0}(u)e_1$ is a tangent vector field of $\mathcal{M}_n(\mathbb{R})$ and*

$$\gamma_t = gZ_{j,0}(u)e_1 \quad (2.11)$$

is a flow equation on $\mathcal{M}_n(\mathbb{R})$.

We call (2.11) the j -th central affine curve flow.

A direct computation shows that the first column of $Z_{2,0}(u)$ is

$$\left(-\frac{2}{n}u_{n-1}, 0, 1, 0, \dots, 0\right)^t.$$

So the central affine curve flow (1.4) can be written as $\gamma_t = gZ_{2,0}(u)e_1$.

Example 2.6. [Higher order central affine curve flows]

We use (2.1) and (2.3) to compute $Z_{j,0}(u)$ and see that the fourth and the fifth central affine curve flows on $\mathcal{M}_3(\mathbb{R})$ are

$$\begin{aligned} \gamma_t &= -\frac{1}{9}(2u_2'' - 3u_1' - 2u_2^2)\gamma + \frac{1}{3}(u_2' - u_1)\gamma_x - \frac{u_2}{3}\gamma_{xx}, \\ \gamma_t &= \frac{1}{9}(-u_1'' + u_1u_2)\gamma - \frac{1}{9}(u_2'' - 3u_1' + u_2^2)\gamma_x + \frac{1}{3}(u_2' - 2u_1)\gamma_{xx}. \end{aligned}$$

For $n \neq 3$, the third central affine curve flow on $\mathcal{M}_n(\mathbb{R})$ is the flow (1.5).

Next we use Theorem 2.3 to define an operator P_u that will be used in the construction of bi-Hamiltonian structure later.

Definition 2.7. Fix $u \in C^\infty(\mathbb{R}, V_n)$, let $V_n^t = \bigoplus_{i=1}^{n-1} \mathbb{R}e_{ni}$, and let

$$P_u : C^\infty(\mathbb{R}, V_n^t) \rightarrow C^\infty(\mathbb{R}, sl(n, \mathbb{R}))$$

denote the map defined by $P_u(v) = C$, where C is the unique $sl(n, \mathbb{R})$ -valued map satisfies $\pi_0(C) = v$ and $[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)$, where π_0 is the canonical projection defined by

$$\pi_0 : sl(n, \mathbb{C}) \rightarrow V_n^t, \quad \pi_0(y) = \sum_{i=1}^{n-1} y_{ni}e_{ni}, \quad \text{for } y = (y_{ij}).$$

It follows from Theorem 2.3 that the entries of $P_u(v)$ are differential polynomials in u and v .

The following is a consequence of Corollary 2.4.

Corollary 2.8. *Suppose u, g are the central affine curvature and moving frame along $\gamma \in \mathcal{M}_n(\mathbb{R})$ respectively. Let $v \in C^\infty(\mathbb{R}, V_n^t)$. Then there exists $\delta\gamma$ tangent to $\mathcal{M}_n(\mathbb{R})$ at γ such that $P_u(v) = g^{-1}\delta g$, where $\delta g = (\delta\gamma, (\delta\gamma)_x, \dots, (\delta\gamma)_x^{(n-1)})$.*

Since $[\partial_x + b + u, Z_{j,0}(u)] \in V_n$, we have $Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u)))$. It follows from $\zeta_j(u) \in \mathcal{N}_n^+$ and $Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u)$ that we have the following.

Corollary 2.9. *Let $\pi_0 : sl(n, \mathbb{R}) \rightarrow V_n^t$ be the canonical projection, $Y(u, \lambda)$ the solution of (2.1), and $Y_{j,0}(u), Z_{j,0}(u, \lambda) = Y_{j,0}(u, \lambda) - \zeta_j(u)$ as in (2.2) and (2.4). Then $Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u))) = P_u(\pi_0(Y_{j,0}(u)))$, and the j -th $A_{n-1}^{(1)}$ -KdV flow (2.5) on $C^\infty(\mathbb{R}, V_n)$ can be written as*

$$u_{t_j} = [\partial_x + b + u, P_u(\pi_0(Y_{j,0}(u)))]. \quad (2.12)$$

Next we identify $T\mathcal{M}(\mathbb{R})_\gamma$ as $C^\infty(\mathbb{R}, \mathbb{R}^{n-1})$.

Corollary 2.10. *The vector field $\sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at $\gamma \in \mathcal{M}_n(\mathbb{R})$ if and only if $\xi_0 = \phi_n(u, \xi_1, \dots, \xi_{n-1})$, where u is the central affine curvature along γ and ϕ_n is the differential polynomial in Theorem 2.3 (ii).*

Proof. Given $\gamma \in \mathcal{M}_n(\mathbb{R})$, let $y : (-\epsilon, \epsilon) \rightarrow \mathcal{M}_n(\mathbb{R})$ with $y(0) = \gamma$. Let $g(\cdot, s)$ and $u(\cdot, s)$ denote the central affine moving frame and curvature along $y(s)$ for each $s \in (-\epsilon, \epsilon)$. Let $\delta\gamma = \frac{dy}{ds}|_{s=0}$, and $\delta g = (\delta\gamma, (\delta\gamma)_x, \dots, (\delta\gamma)_x^{(n-1)})$. Take s derivative of $g^{-1}g_x = b + u$ to see that $[\partial_x + b + u, g^{-1}\delta g] \in V_n$, i.e., $C = (C_{ij}) := g^{-1}\delta g$ satisfies (2.8). By Theorem 2.3, we have $\delta\gamma = gCe_1 = \sum_{i=1}^n C_{i1}\gamma_x^{(i-1)}$ and $C_{11} = \phi_n(u, C_{21}, \dots, C_{n1})$. \square

Example 2.11. The proof of Theorem 2.3 gives an algorithm to compute the formula for ϕ_n . For example, we get $\sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $T\mathcal{M}_n(\mathbb{R})$ at γ if

$$\xi_0 = -\frac{1}{2}\xi_1', \quad \text{for } n = 2,$$

$$\xi_0 = -\frac{1}{3}(\xi_2'' + 3\xi_1' + 2u_2\xi_2), \quad \text{for } n = 3,$$

$$\xi_0 = -\frac{1}{4}(\xi_3''' + 4\xi_2'' + 6\xi_1' + 3u_3'\xi_3 + 5u_3\xi_3' + 2u_2\xi_3 + 2u_3\xi_2), \quad \text{for } n = 4.$$

For $n = 5$, we have

$$\begin{aligned} \xi_0 = & -\frac{1}{5}(\xi_4^{(4)} + 5\xi_3^{(3)} + 10\xi_2'' + 10\xi_1' + 6u_3'\xi_4 + 9u_3\xi_4' + 4(u_4\xi_4)'' + 3(u_4\xi_4)') \\ & + 2u_4\xi_4'' + 3(u_4\xi_3)' + 4u_4\xi_3' + 4u_2\xi_4 + 3u_3\xi_3 + 2u_4\xi_2 + 2u_4^2\xi_4). \end{aligned}$$

Here we use $y' = y_x$, $y'' = y_x^{(2)}$, $y^{(3)} = y_x^{(3)}$ etc. For general n , we have

$$\xi_0 = -\frac{1}{n}((\xi_{n-1})_x^{(n-1)} + \dots).$$

3. CENTRAL AFFINE CURVE FLOWS AND THE $A_{n-1}^{(1)}$ -KdV HIERARCHY

In this section, we prove the following results:

- (1) The affine curvature map $\Psi(\gamma) = u$ gives a one to one correspondence between the space of solutions of the j -th central affine curve flow

(2.11) on $\mathbb{R}^n \setminus \{0\}$ modulo $SL(n, \mathbb{R})$ and the space of solutions of the j -th $A_{n-1}^{(1)}$ -KdV flow,

$$u_t = [\partial_x + b + u, Z_{j,0}(u)]. \quad (3.1)$$

When $n = 3$, this results was obtained in [3].

- (2) We use solutions of the Cauchy problem for the j -th $A_{n-1}^{(1)}$ -KdV flow to solve the Cauchy problem for (2.11) with periodic initial data or initial data having rapidly decaying central affine curvatures.

Theorem 3.1. *Let $u = \sum_{i=1}^{n-1} u_i e_{in}$ be a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (3.1), and $c_0 \in SL(n, \mathbb{R})$ a constant. Let $g : \mathbb{R}^2 \rightarrow SL(n, \mathbb{R})$ denote the solution of*

$$g^{-1}g_x = b + u, \quad g^{-1}g_t = Z_{j,0}(u), \quad (3.2)$$

with $g(0,0) = c_0$. Then $\gamma(x,t) := g(x,t)e_1$ is a solution of the j -th central affine curve flow (2.11) with central affine curvature $u(x,t)$.

Proof. Note that $g^{-1}g_x = b + u$ implies $g = (\gamma, \gamma_x, \dots, \gamma_x^{(n-1)})$ and $\gamma_x^{(n)} = u_1\gamma + \dots + u_{n-1}\gamma_x^{(n-2)}$. So $\gamma(\cdot, t) \in \mathcal{M}_n(\mathbb{R})$ and u_1, \dots, u_{n-1} are the central affine curvatures of $\gamma(\cdot, t)$. Since $g_t = gZ_{j,0}(u)$, we get $\gamma_t = g_te_1 = gZ_{j,0}(u)e_1$, which is the j -th central affine curve flow. \square

The converse is also true.

Theorem 3.2. *Let γ be a solution of (2.11) on $\mathcal{M}_n(\mathbb{R})$, and $u(\cdot, t) = \sum_{i=1}^{n-1} u_i(\cdot, t)e_{in}$ the central affine curvature along $\gamma(\cdot, t)$. Then u is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (3.1).*

Proof. Let $g(\cdot, t) = (\gamma, \gamma_x, \dots, \gamma_x^{(n-1)})(\cdot, t)$ be the central affine moving frame for $\gamma(\cdot, t) \in \mathcal{M}_n(\mathbb{R})$. Then $g^{-1}g_x = b + u$. Next we compute g_t . Let η_i denote the i -th column of $gZ_{j,0}(u)$. Then (2.11) is $\gamma_t = \eta_1$. Since $[\partial_x + b + u, Z_{j,0}(u)] \in V_n$, Theorem 2.3 implies that $\eta_i = (\eta_1)_x^{(i-1)}$. A direct computation implies that $(\gamma_x^{(i-1)})_t = (\gamma_t)_x^{(i-1)} = (\eta_1)_x^{(i-1)}$ for $2 \leq i \leq n$. This proves that $g_t = gZ_{j,0}(u)$. It follows from Proposition 2.1 that u is a solution of (3.1). \square

Corollary 3.3. *Let Ψ denote the central affine curvature map, and γ_1, γ_2 solutions of (2.11) on $\mathcal{M}_n(\mathbb{R})$. Then*

- (1) $\Psi(\gamma_1(\cdot, t)) = \Psi(\gamma_2(\cdot, t))$ if and only if there is a constant c_0 in $SL(n, \mathbb{R})$ such that $\gamma_2 = c_0\gamma_1$,
- (2) Ψ induces a bijection between the space of solutions of (2.11) modulo $SL(n, \mathbb{R})$ and the space of solutions of (3.1).

Next we discuss the Cauchy problem for the j -th central affine curve flow (2.11). The Cauchy problem for the $A_{n-1}^{(1)}$ -KdV hierarchy (2.12) is solved for an open dense subset of rapidly decaying smooth initial data using the method of inverse scattering (cf. [2]). As a consequence, we get the solution for the Cauchy problem for the curve flow (2.11):

Theorem 3.4. [Cauchy problem with rapidly decaying affine curvatures]
 Let $j \geq 1$ and $j \not\equiv 0 \pmod{n}$. Given $\gamma_0 \in \mathcal{M}_n(\mathbb{R})$ with rapidly decaying central affine curvatures u_1^0, \dots, u_{n-1}^0 , let g_0 be the central affine moving frame along γ_0 . Suppose $u = \sum_{i=1}^{n-1} u_i e_{in}$ is the solution of the j -th flow (3.1) in the $A_{n-1}^{(1)}$ -KdV hierarchy with $u(x, 0) = \sum_{i=1}^{n-1} u_i^0(x) e_{in}$. Let $g(x, t) : \mathbb{R}^2 \rightarrow SL(n, \mathbb{R})$ be the solution of (3.2) with initial data $g(0, 0) = g_0(0)$. Then $\gamma = g e_1$ is the solution of the j -th central affine curve flow (2.11) with $\gamma(x, 0) = \gamma_0(x)$. Moreover, the central affine curvatures of $\gamma(\cdot, t)$ are also rapidly decaying.

Finally, we use the solution of Cauchy problem of the second $A_{n-1}^{(1)}$ -KdV flow with periodic initial data to solve the Cauchy problem for the curve flow (1.4) with periodic initial data. By Theorem 3.1, we only need to solve the period problem of (3.2). In fact, we have the following.

Theorem 3.5. [Cauchy Problem with periodic initial data]
 Let $\gamma_0 \in \mathcal{M}_n(S^1)$, and u_1^0, \dots, u_{n-1}^0 the central affine curvatures of γ_0 . Suppose $u = \sum_{i=1}^{n-1} u_i e_{in}$ is the solution of the periodic Cauchy problem of (3.1) with initial data $u(x, 0) = \sum_{i=1}^{n-1} u_i^0 e_{in}$. Let $g : \mathbb{R}^2 \rightarrow SL(n, \mathbb{R})$ be the solution of (3.2) with initial data $c_0 = g_0(0)$, where g_0 is the central affine frame along γ_0 . Then $\gamma = g e_1$ is a solution of (2.11) with initial data $\gamma(x, 0) = \gamma_0(x)$. Moreover, $\gamma(x, t)$ is periodic in x and $\{u_i(\cdot, t), 1 \leq i \leq n-1\}$ are the central affine curvatures for $\gamma(\cdot, t)$.

Proof. Note that both g_0 and $g(\cdot, 0)$ satisfy the same ordinary differential equation, $g^{-1} g_x = b + u(x, 0)$, and have the same initial data. So the uniqueness of ordinary differential equations implies that $g(x, 0) = g_0(x)$. It follows from Theorem 3.1 that $\gamma(x, t) = g(x, t) e_1$ is a solution of the curve flow (2.11). Moreover, $\gamma(x, 0) = g(x, 0) e_1 = \gamma_0(x)$. It remains to prove that γ is periodic in x .

Since γ_0 is periodic with period 2π , g_0 and u_0 are periodic in x with period 2π . Since $u(x, t)$ is periodic in x , so is $Z_{j,0}(u)$. It suffices to prove

$$y(t) = g(2\pi, t) - g(0, t)$$

is identically zero. To do this, we calculate

$$\begin{aligned} y_t &= g_t(2\pi, t) - g_t(0, t) \\ &= (g Z_{j,0}(u))(2\pi, t) - (g Z_{j,0}(u))(0, t) = (g(2\pi, t) - g(0, t)) Z_{j,0}(u(0, t)) \\ &= y(t) Z_{j,0}(u(0, t)). \end{aligned}$$

Since g_0 is periodic in x with period 2π , $y(0) = g(2\pi, 0) - g(0, 0) = 0$. Note that $Z_{j,0}(u(0, t))$ is given and 0 is the solution of $y_t = y Z_{j,0}(u(0, t))$ with the same initial condition $y(0) = 0$. So it follows from the uniqueness of ordinary differential equations that y is identically zero. \square

4. BÄCKLUND TRANSFORMATIONS

In this section, we use Bäcklund transformations (BTs) for the $A_{n-1}^{(1)}$ -KdV hierarchy given in [13] to construct BTs for the j -th central affine curve flow (2.11), a permutability formula, and infinitely many explicit rational and soliton solutions of the curve flow (2.11).

We first summarize results concerning BTs of the $A_{n-1}^{(1)}$ -KdV hierarchy obtained in [13]. Let E and \tilde{E} be frames of solutions $u = \sum_{i=1}^{n-1} u_i e_{in}$ and $\tilde{u} = \sum_{i=1}^n \tilde{u}_i e_{in}$ of the j -th $A_{n-1}^{(1)}$ -KdV flow (3.1) respectively. Suppose $\tilde{E} = Ef$ and the first column of $f(x, t, \lambda)$ is $(h, 1, 0, \dots)^t$ for some $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$. Then there exist differential polynomials $s_i(u, h)$, $\eta_{n,j}(u, h)$, and $\xi_n(u, h)$ such that

$$(i) \quad \tilde{u}_i = u_i + s_i(u, h), \quad 1 \leq i \leq n-1, \quad (4.1)$$

(ii) f is determined by h , in fact, $f = J + hI_n + N(u, h)$, where N is strictly upper triangular and the ij -th entry of $N(u, h)$ is

$$\begin{cases} N_{ij}(u, h) = C_{j-1, i-1} h_x^{(j-i)}, & 1 \leq i < j < n, \\ N_{in} = u_i + s_i(u, h) + C_{n-1, i-1} h_x^{(n-i)}, & 1 \leq i \leq n-1, \end{cases}$$

and $C_{j,i} = \frac{j!}{i!(j-i)!}$. Henceforth we use $f_{u,h}$ to denote such f .

(iii) $\det(f_{u,h}) = (-1)^{n-1}(\lambda + h_x^{(n-1)} - \xi_n(u, h))$,

(iv) there exists a constant $k \in \mathbb{C}$ such that h satisfies

$$(BT)_{u,k} \quad \begin{cases} h_x^{(n-1)} = \xi_n(u, h) - k, \\ h_t = \eta_{n,j}(u, h). \end{cases} \quad (4.2)$$

Theorem 4.1. ([13]) *Suppose E is a frame of a solution $u = \sum_{i=1}^{n-1} u_i e_{in}$ of the j -th $A_{n-1}^{(1)}$ -KdV flow (3.1) such that $E(x, t, \lambda)$ is holomorphic for λ in an open subset \mathcal{O} of \mathbb{C} . Let $k \in \mathcal{O}$, $\mathbf{c}_0 = (c_1, \dots, c_{n-1}, -1)^t$ a constant vector in \mathbb{C}^n , $(v_1, \dots, v_n)^t := E(\cdot, \cdot, k)^{-1}(\mathbf{c}_0)$, and $h := -\frac{v_{n-1}}{v_n}$. Then we have the following:*

- (1) h is a solution of $(BT)_{u,k}$, i.e., (4.2).
- (2) All solutions of (4.2) are obtained this way.
- (3) \tilde{u} defined by (4.1) is a solution of (3.1), (we will denote \tilde{u} as $h * u$).
- (4) $\det(f_{u,h}) = (-1)^{n-1}(\lambda - k)$.
- (5) $\tilde{E} = Ef_{u,h}^{-1}$ is a frame of \tilde{u} and $\tilde{E}(x, t, \lambda)$ is holomorphic for $\lambda \in \mathcal{O} \setminus \{k\}$.
- (6) Let $C(\lambda) = e_{1n}(\lambda - k) + b + \sum_{i=1}^{n-1} c_i e_{i+1, n}$. Then

$$\tilde{E}(x, t, \lambda) = C(\lambda)E(x, t, \lambda)f_{u,h}(x, t, \lambda)^{-1}$$

is a frame of \tilde{u} that is holomorphic for all $\lambda \in \mathcal{O}$.

As a consequence of Theorems 3.1, 3.2, and 4.1, we obtain BTs for the j -th central affine flow (2.11):

Theorem 4.2. [BT for the j -th central affine curve flow with $k \neq 0$]

Let $\gamma(x, t)$ be a solution of the j -th central affine curve flow (2.11) on $\mathcal{M}_n(\mathbb{R})$, $g(\cdot, t)$ and $u(\cdot, t)$ the central affine moving frame and curvature of $\gamma(\cdot, t)$ respectively. Let k be a non-zero real constant, and $d(k) \in GL(n, \mathbb{R})$ such that $\det(d(k)) = (-1)^n k$. Let h be a solution of $(\text{BT})_{u,k}$ (i.e., (4.2)), and

$$\tilde{g}(x, t) = d(k)g(x, t)f_{u,h}(x, t, 0)^{-1}.$$

Then $h * \gamma(x, t) = \tilde{g}(x, t)e_1$ is again a solution of (2.11) and $\tilde{g}(\cdot, t)$ is the central affine moving frame along $h * \gamma(\cdot, t)$.

Proof. By Theorem 3.2, u is a solution of (3.1). Since h is a solution of $(\text{BT})_{u,k}$, by Theorem 4.1 (4) we have $\det(f_{u,h}) = (-1)^{n-1}(\lambda - k)$ and $\hat{g}(x, t) = g(x, t)f_{u,h}^{-1}(x, t, 0)$ is a parallel frame of the Lax pair for \tilde{u} at $\lambda = 0$. So $\det(\hat{g}) = \det(gf_{u,h}^{-1}(x, t, 0)) = (-1)^n k^{-1}$. Since $d(k)$ is a constant matrix, $\tilde{g} = d(k)\hat{g}$ is a frame of the Lax pair of \tilde{u} at $\lambda = 0$ with determinant 1. Then this theorem follows from Theorems 3.1 and 4.1. \square

Given an $n \times n$ matrix M , we use M^\sharp to denote the *cofactor matrix* of M , i.e., the ij -th entry of M^\sharp is equal to $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by crossing out the j -th row and the i -th column of M . Then $MM^\sharp = \det(M)I_n$.

Theorem 4.3. [BT for the j -th central affine curve flow with $k = 0$]

Let γ , u , and g be as in Theorem 4.2, and $E(x, t, \lambda)$ the frame for the solution u of (3.1) satisfying $E(0, 0, 0) = g(0, 0)$. Let $E_1(x, t) = \frac{\partial}{\partial \lambda}|_{\lambda=0} E(x, t, \lambda)$, $\mathbf{c}_0 = (c_1, \dots, c_{n-1}, -1)^t$ a constant in \mathbb{R}^n , $(v_1, v_2, \dots, v_n)^t := g^{-1}\mathbf{c}_0$, and $h = -\frac{v_{n-1}}{v_n}$. Then

$$\tilde{\gamma} = (e_{1n}g + AE_1)f_{u,h}^\sharp(x, t, 0)e_1$$

is a solution of (2.11), where $A = b + \sum_{i=1}^{n-1} c_i e_{i+1,n}$ and $f_{u,h}^\sharp$ is the cofactor matrix of $f_{u,h}$.

Proof. Since both $E(x, t, 0)$ and $g(x, t)$ are solutions of

$$\begin{cases} g^{-1}g_x = b + u, \\ g^{-1}g_t = Z_j(u, 0), \end{cases}$$

with the same initial data at $(0, 0)$, we have $E(x, t, 0) = g(x, t)$.

Let $C(\lambda) = J + \sum_{i=1}^{n-1} c_i e_{i+1,n}$. By Theorem 4.1,

$$F(x, t, \lambda) = C(\lambda)E(x, t, \lambda)f_{u,h}^{-1}(x, t, \lambda) = \frac{(-1)^{n-1}}{\lambda} C(\lambda)E(x, t, \lambda)f_{u,h}^\sharp(x, t, \lambda)$$

is a frame for $h * u$ and is holomorphic at $\lambda = 0$. So

$$F(x, t, 0) = (-1)^{n-1} \frac{\partial}{\partial \lambda}|_{\lambda=0} CEf_{u,h}^\sharp. \quad (4.3)$$

By (4.3), we get

$$F(x, t, 0) = (e_{1n}g + AE_1)f_{u,h}^\sharp(x, t, 0) + Ag \left(\frac{\partial}{\partial \lambda} \Big|_{\lambda=0} f_{u,h}^\sharp \right).$$

But the first column of $f_{u,h}^\sharp$ is independent of λ , hence the last term in the above formula is zero and

$$F(x, t, 0) = (e_{1n}g + AE_1)f_{u,h}^\sharp(x, t, 0).$$

Note that $\det(E(x, t, 0)) = \det(g(x, t)) = 1$. By Theorem 4.1, $\det(f_{u,h}) = (-1)^{n-1}\lambda$. Note that $\det(C(\lambda)) = (-1)^{n-1}\lambda$. Hence $\det(F(x, t, 0)) = 1$. By Theorem 3.1, $F(\cdot, \cdot, 0)e_1$ is a solution of the curve flow (2.11). \square

Theorem 4.4. ([13]) [Permutability for BT of the $A_{n-1}^{(1)}$ -KdV hierarchy] *Let u be a solution of (3.1), $k_1, k_2 \in \mathbb{C}$ constants, h_i solutions of $(\text{BT})_{u,k_i}$ (4.2), and $h_i * u$ the solution of (3.1) constructed from u and h_i for $i = 1, 2$ as in Theorem 4.1. Suppose $h_1 \neq h_2$. Set*

$$\begin{cases} \tilde{h}_1 = h_1 + \frac{(h_1 - h_2)_x}{h_1 - h_2}, \\ \tilde{h}_2 = h_2 + \frac{(h_1 - h_2)_x}{h_1 - h_2}. \end{cases}$$

Then

- (i) \tilde{h}_1 is a solution of $(\text{BT})_{h_2 * u, k_1}$ and \tilde{h}_2 is a solution of $(\text{BT})_{h_1 * u, k_2}$,
- (ii) $\tilde{h}_1 * (h_2 * u) = \tilde{h}_2 * (h_1 * u)$.

As a consequence of Theorems 3.1 and 4.4, we obtain the following.

Theorem 4.5. [Permutability for the j -th central affine curve flow]

Let γ , u , g , and $d(k)$ be as in Theorem 4.2, $k_1, k_2, h_1, h_2, \tilde{h}_1, \tilde{h}_2$, as in Theorem 4.4, and

$$g_i(x, t) = d(k_i)g(x, t)f_{u,h_i}(x, t, 0)^{-1}, \quad \gamma_i = g_i e_1$$

*for $i = 1, 2$, where $e_1 = (1, 0, \dots, 0)^t$. Then $\tilde{h}_2 * \gamma_1 = \tilde{h}_1 * \gamma_2$ is again a solution of (2.11), i.e.,*

$$d(k_2)g_1(x, t)f_{u,\tilde{h}_2}(x, t, 0)^{-1}e_1 = d(k_1)g_2(x, t)f_{u,\tilde{h}_1}(x, t, 0)^{-1}e_1$$

is again a solution of (2.11).

Next we write down formulas of BTs for the second central affine curve flow on $\mathcal{M}_3(\mathbb{R})$ in explicit forms. The second central affine curve flow on $\mathcal{M}_3(\mathbb{R})$ is

$$\gamma_t = -\frac{2}{3}u_2\gamma + \gamma_{xx}, \tag{4.4}$$

and the second $A_2^{(1)}$ -KdV flow is

$$\begin{cases} (u_1)_t = (u_1)_{xx} - \frac{2}{3}(u_2)_{xxx} + \frac{2}{3}u_2(u_2)_x, \\ (u_2)_t = -(u_2)_{xx} + 2(u_1)_x. \end{cases} \tag{4.5}$$

System (BT)_{u,k} for the second flow (4.5) is

$$\begin{cases} h_{xx} = -u_1 + (u_2)_x + hu_2 - 3hh_x - h^3 - k, \\ h_t = \frac{2}{3}(u_2)_x - h_{xx} - 2hh_x. \end{cases} \quad (4.6)$$

and the new solution $h * u = \tilde{u}_1 e_{13} + \tilde{u}_2 e_{23}$ is given by

$$\begin{cases} \tilde{u}_1 = u_1 - (u_2)_x + 3hh_x, \\ \tilde{u}_2 = u_2 - 3h_x. \end{cases} \quad (4.7)$$

Moreover, if E is a frame for u , then $E f_{u,h}^{-1}$ is a frame for $h * u$, where

$$f_{u,h}(x, t, \lambda) = e_{13}\lambda + \begin{pmatrix} h & h_x & u_1 - (u_2)_x + h_{xx} + 3hh_x \\ 1 & h & u_2 - h_x \\ 0 & 1 & h \end{pmatrix}.$$

We choose $d(k) = -k^{1/3}I_3$ in Theorem 4.2 for $n = 3$. Then we obtain:

Corollary 4.6. *Let γ be a solution of (4.4) on $\mathcal{M}_3(\mathbb{R})$, $u = \sum_{i=1}^2 u_i e_{i3}$ the central affine curvature of $\gamma(\cdot, t)$, k a non-zero real constant, and h a solution of (4.6). Then*

$$h * \gamma = k^{-2/3}((h^2 + h_x - u_2)\gamma - h\gamma_x + \gamma_{xx})$$

is a solution of (4.4) with central affine curvature \tilde{u}_1, \tilde{u}_2 given by (4.7).

Corollary 4.7. *Let γ and u be as in Corollary 4.6, $g(\cdot, t)$ the central affine moving frame along $\gamma(\cdot, t)$, and $E(x, t, \lambda)$ the frame of u with $E(0, 0, \lambda) = g(0, 0)$. Let $E_1(x, t) = \frac{\partial}{\partial \lambda}|_{\lambda=0} E(x, t, \lambda)$, $c_1, c_2 \in \mathbb{R}$, $A = e_{21} + e_{32} + \sum_{i=1}^2 c_i e_{i+1,3}$, $v = (v_1, v_2, v_3)^t = g^{-1}(c_1, c_2, -1)^t$, and $h = -\frac{v_2}{v_3}$. Then*

- (i) $\tilde{\gamma} = (e_{13}g + AE_1)(h^2 + h_x - u_2, -h, 1)^t$ is a solution of (4.4),
- (ii) $\tilde{E}(x, t, \lambda) = (\lambda e_{13} + A)E(x, t, \lambda)f_{u,h}^{-1}$ is a frame for $\tilde{u} = h * u$.

Note that $[\partial_x + J, \partial_t + J^j] = 0$ implies that $u = 0$ is a solution of the j -th $A_{n-1}^{(1)}$ -KdV flow (3.1) and $E(x, t, \lambda) = \exp(xJ + tJ^j)$ is a frame of $u = 0$. Hence

$$\gamma(x, t) = \exp(bx + b^j t)e_1 \quad (4.8)$$

is a solution of the j -th central affine curve flow (2.11) with zero central affine curvatures and $g(x, t) = \exp(bx + b^j t)$ is the central affine moving frame of γ . So we can apply Theorem 4.2 and 4.4 repeatedly to obtain an infinitely many families of explicit solutions of (2.11).

Example 4.8 (Explicit 1-soliton solutions). We apply BT to the solution $u = 0$ of the second $A_2^{(1)}$ -KdV flow (4.5) with $k = 8\mu^3 \in \mathbb{C}$ (for detail cf. [13]) to obtain 1-soliton complex valued solutions

$$\begin{cases} u_1 = 9\mu^3 \text{sech}^2(\sqrt{3}\mu(x - 2\mu it))(\sqrt{3} \tanh(\sqrt{3}\mu(x - 2\mu it)) + i), \\ u_2 = -9\mu^2 \text{sech}^2(\sqrt{3}\mu(x - 2\mu it)) \end{cases}$$

for (4.5). Apply BT to $u = 0$ with $k = -c^3$ with constant $c \in \mathbb{R} \setminus 0$ to get the following real solutions of (4.5),

$$\begin{cases} u_1 = 9c^3 \sec^2(\sqrt{3}c(x + 2ct))(1 + \sqrt{3} \tan(\sqrt{3}c(x + 2ct))), \\ u_2 = 9c^2 \sec^2(\sqrt{3}c(x + 2ct)). \end{cases}$$

We have seen that $\gamma(x, t) = (1, x, \frac{1}{2}x^2 + t)^t$ is a solution of (4.4) with zero central affine curvatures. So we apply Theorem 4.2 and obtain the following solutions of the second central affine curve flow (4.4):

$$\tilde{\gamma} = \begin{pmatrix} 2(\xi - 1) \\ 2x(\xi - 1) + c^{-1}(\xi + 1) \\ (x^2 + 2t)(\xi - 1) + c^{-1}x(\xi + 1) + c^{-2} \end{pmatrix},$$

where $c \in \mathbb{R} \setminus 0$ is a constant and $\xi(x, t) = \sqrt{3} \tan(\sqrt{3}c(x + 2ct))$.

Example 4.9. [Rational solutions for $n = 3$]

We apply BT with parameter $k = 0$ to the vacuum solution $u = 0$ of (4.5). Recall that $\gamma(x, t) = (1, x, \frac{x^2}{2} + t)^t$ is a solution of (4.4) with zero central affine curvatures and $E(x, t, \lambda) = e^{xJ+tJ^2}$ is a frame of the trivial solution of (4.5). The constant term of $E(x, t, \lambda)$ as a power series in λ is

$$g(x, t) = E_0(x, t) = \exp(bx + b^2t) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ \frac{x^2}{2} + t & x & 1 \end{pmatrix},$$

which is the central affine moving frame along γ . A direct computation implies that the coefficient of λ of $E(x, t, \lambda)$ is

$$E_1(x, t) = \begin{pmatrix} xt + \frac{1}{6}x^3 & t + \frac{1}{2}x^2 & x \\ \frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4 & xt + \frac{1}{6}x^3 & t + \frac{1}{2}x^2 \\ \frac{1}{2}xt^2 + \frac{1}{6}x^3t + \frac{1}{5!}x^5 & \frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4 & xt + \frac{1}{6}x^3 \end{pmatrix}.$$

We apply Theorem 4.3 to $u = 0$ and $v_0 = (a_1, a_2, 1)^t$ to see that

$$\tilde{\gamma} = \begin{pmatrix} (h^2 + h_{xx})(\frac{x^2}{2} + t) - hx + 1 \\ (h^2 + h_{xx})(\frac{1}{6}x^3 + xt) - h(\frac{1}{2}x^2 + t) + x \\ (h^2 + h_{xx})(\frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4) - h(\frac{1}{6}x^3 + xt) + \frac{1}{2}x^2 + t \end{pmatrix}$$

is a rational solution of (4.4), where

$$h = \frac{a_1x - a_2}{1 + a_1(\frac{x^2}{2} - t) - a_2x}.$$

The central affine curvatures for $\tilde{\gamma}$ are

$$\begin{cases} u_1 = -\frac{3(a_1x - a_2)(\frac{1}{2}a_1^2x^2 - a_1a_2x + a_1^2t + a_2^2 - a_1)}{(1 + a_1(\frac{x^2}{2} - t) - a_2x)^3}, \\ u_2 = \frac{3(\frac{1}{2}a_1^2x^2 - a_1a_2x + a_1^2t + a_2^2 - a_1)}{(1 + a_1(\frac{x^2}{2} - t) - a_2x)^2}. \end{cases}$$

If we apply Theorem 4.3 or the permutability formula repeatedly, then we obtain infinitely many families of rational solutions for the second central affine curve flow (4.4) on $\mathcal{M}_3(\mathbb{R})$.

5. BI-HAMILTONIAN STRUCTURE FOR THE $A_{n-1}^{(1)}$ -KdV HIERARCHY

We first review the bi-Hamiltonian structure and commuting conservation laws for the $A_{n-1}^{(1)}$ -KdV hierarchy (cf. [4], [5]). Then we write down the formulas for the Poisson operators in terms of the operator P_u defined in Definition 2.7 and compute the kernel (i.e., Casimirs) of these operators.

The group $C^\infty(S^1, N_n^+)$ acts on $C^\infty(S^1, \mathcal{B}_n^+)$ by gauge transformation

$$g * q = g(b + q)g^{-1} - g_x g^{-1} - b,$$

or equivalently,

$$g * (\partial_x + b + q) = g(\partial_x + b + q)g^{-1} = \partial_x + b + g * q.$$

It was proved in [5] (cf. also in [11]) that $C^\infty(S^1, V_n)$ is a cross section of the gauge action $C^\infty(S^1, N_n^+)$ on $C^\infty(S^1, \mathcal{B}_n^+)$, i.e., given $q \in C^\infty(S^1, \mathcal{B}_n^+)$ there exist unique $\Delta \in C^\infty(S^1, N_n^+)$ and $u \in C^\infty(S^1, V_n)$ such that $\Delta * q = u$. In other words, each gauge orbit of $C^\infty(S^1, N_n^+)$ meets $C^\infty(S^1, V_n)$ exactly once. So $C^\infty(S^1, V_n)$ is isomorphic to the orbit space $\frac{C^\infty(S^1, \mathcal{B}_n^+)}{C^\infty(S^1, N_n^+)}$.

Let $\langle \cdot, \cdot \rangle$ be the bi-linear form on $C^\infty(S^1, \mathfrak{sl}(n, \mathbb{R}))$ defined by

$$\langle y_1, y_2 \rangle = \oint \text{tr}(y_1(x)y_2(x))dx. \quad (5.1)$$

The gradient of $\mathcal{F} : C^\infty(S^1, \mathcal{B}_n^+) \rightarrow \mathbb{R}$ at $q \in C^\infty(S^1, \mathcal{B}_n^+)$ is the unique element $\nabla \mathcal{F}(q)$ in $C^\infty(S^1, \mathcal{B}_n^+)$ defined by

$$d\mathcal{F}_q(y) = \langle \nabla \mathcal{F}(q), y \rangle$$

for all $y \in C^\infty(S^1, \mathcal{B}_n^+)$.

The two Poisson structures on $C^\infty(S^1, \mathcal{B}_n^+)$ given in [5] are:

$$\{\mathcal{F}_1, \mathcal{F}_2\}_1(u) = \langle [e_{1n}, \nabla \mathcal{F}_1(u)], \nabla \mathcal{F}_2(u) \rangle, \quad (5.2)$$

$$\{\mathcal{F}_1, \mathcal{F}_2\}_2(u) = \langle [\partial_x + b + u, \nabla \mathcal{F}_1(u)], \nabla \mathcal{F}_2(u) \rangle. \quad (5.3)$$

These two Poisson structures are invariant under the action of $C^\infty(S^1, N_n^+)$. So they induce two Poisson structures on $C^\infty(S^1, V_n)$.

The gradient $\nabla F(u)$ of a functional F on $C^\infty(S^1, V_n)$ at u is the unique element in $C^\infty(S^1, V_n^t)$ satisfying

$$dF_u(v) = \langle \nabla F(u), v \rangle$$

for all $v \in C^\infty(S^1, V_n)$.

Recall that $C^\infty(S^1, V_n)$ is isomorphic to the orbit space $\frac{C^\infty(S^1, \mathcal{B}_n^+)}{C^\infty(S^1, N_n^+)}$. So given a functional F on $C^\infty(S^1, V_n)$, there is a unique $C^\infty(S^1, N_n^+)$ -invariant function \tilde{F} on $C^\infty(S^1, \mathcal{B}_n^+) \rightarrow \mathbb{R}$ whose restriction to $C^\infty(S^1, V_n)$ is F , i.e.,

$\tilde{F}(q) = F(u)$ if $u \in C^\infty(S^1, V_n)$ lies in the same $C^\infty(S^1, N_n^+)$ -orbit as q . The following Proposition gives the relation between $\nabla \tilde{F}(u)$ and $\nabla F(u)$ for $u \in C^\infty(S^1, V_n)$.

Proposition 5.1. *Let F be a functional on $C^\infty(S^1, V_n)$, and \tilde{F} the functional on $C^\infty(S^1, \mathcal{B}_n^+)$ invariant under $C^\infty(S^1, N_n^+)$ defined by F . Then*

$$\nabla \tilde{F}(u) = P_u(\nabla F(u)),$$

where $u \in C^\infty(S^1, V_n)$ and P_u is the operator defined Definition 2.7.

Proof. Note that the infinitesimal vector field $\tilde{\xi}$ defined by ξ in $C^\infty(S^1, \mathcal{N}_n^-)$ for the gauge action is

$$\tilde{\xi}(q) = -[\partial_x + b + q, \xi].$$

where $q \in C^\infty(S^1, \mathcal{B}_n^+)$. By assumption, $\tilde{F}(f * q) = \tilde{F}(q)$ for all $q \in C^\infty(S^1, \mathcal{B}_n^+)$ and $f \in C^\infty(S^1, N_n^+)$. So $d\tilde{F}_q(\tilde{\xi}(q)) = 0$. But

$$d\tilde{F}_q(\tilde{\xi}(q)) = \langle \nabla \tilde{F}(q), \tilde{\xi}(q) \rangle = -\langle \nabla \tilde{F}(q), [\partial_x + b + q, \xi] \rangle = \langle [\partial_x + b + q, \nabla \tilde{F}(q)], \xi \rangle$$

for all $\xi \in C^\infty(S^1, \mathcal{N}_n^+)$. So

$$[\partial_x + b + u, \nabla \tilde{F}(q)] \in C^\infty(S^1, \mathcal{B}_n^-).$$

To prove $\nabla \tilde{F}(u) = P_u(\nabla F(u))$ for $u \in C^\infty(S^1, V_n)$ is equivalent to prove

$$d\tilde{F}_u(y) = \langle P_u(\nabla F(u)), y \rangle \quad (5.4)$$

for all $y \in C^\infty(S^1, \mathcal{B}_n^+)$.

We first prove (5.4) for $y \in C^\infty(S^1, V_n)$. Given $u, v \in C^\infty(S^1, V_n)$, we have

$$dF_u(v) = \langle \nabla F(u), v \rangle = d\tilde{F}_u(v) = \langle \nabla \tilde{F}(u), v \rangle.$$

So $\langle \nabla F(u) - \nabla \tilde{F}(u), v \rangle = 0$ for all $v \in C^\infty(S^1, V_n)$. This implies that

$$\pi_0(\nabla \tilde{F}(u)) = \nabla F(u),$$

where $\pi_0 : sl(n, \mathbb{R}) \rightarrow V_n^t$ is the canonical projection. By definition of P_u , we have $\pi_0(P_u(\nabla F(u))) = \nabla F(u)$. So we obtain $d\tilde{F}_u(v) = dF_u(v) = \langle P_u(\nabla F(u)), v \rangle$, i.e., (5.4) is true for $y \in C^\infty(S^1, V_n)$.

Since $C^\infty(S^1, V_n)$ is a cross section of the gauge action of $C^\infty(S^1, N_n^+)$ on $C^\infty(S^1, \mathcal{B}_n^+)$, the tangent space of $C^\infty(S^1, \mathcal{B}_n^+)$ at $u \in C^\infty(S^1, V_n)$ can be written as a direct sum of $C^\infty(S^1, V_n)$ and the tangent space of the $C^\infty(S^1, N_n^+)$ -orbit at u . Since \tilde{F} is invariant under $C^\infty(S^1, N_n^+)$, we have $d\tilde{F}_u(\tilde{\xi}(u)) = 0$ for all $\xi \in C^\infty(S^1, \mathcal{N}_n^+)$. So

$$\begin{aligned} \langle P_u(\nabla F(u)), \tilde{\xi}(u) \rangle &= \langle P_u(\nabla F(u)), -[\partial_x + b + u, \xi] \rangle \\ &= \langle [\partial_x + b + u, P_u(\nabla F(u))], \xi \rangle. \end{aligned}$$

By definition of P_u , we have $[\partial_x + b + u, P_u(\nabla F(u))] \in C^\infty(S^1, V_n)$. Since $\xi \in C^\infty(S^1, \mathcal{N}_n^+)$, we conclude that $\langle P_u(\nabla F(u)), \tilde{\xi}(u) \rangle = 0$. This proves $d\tilde{F}_u(\tilde{\xi}(u)) = \langle P_u(\nabla F(u)), \tilde{\xi}(u) \rangle = 0$. So (5.4) is true for y in the tangent space of $C^\infty(S^1, N_n^+)$ -orbit at u . This completes the proof. \square

So the Poisson structures on $C^\infty(S^1, V_n)$ induced from (5.2) and (5.3) are given as follows:

$$\begin{aligned}\{F_1, F_2\}_1(u) &= \langle [e_{1n}, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle, \\ \{F_1, F_2\}_2(u) &= \langle [\partial_x + b + u, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle,\end{aligned}$$

where $P_u : C^\infty(S^1, V_n^t) \rightarrow C^\infty(S^1, sl(n, \mathbb{R}))$ is defined in Definition 2.7.

Let

$$(J_i)_u : C^\infty(S^1, V_n^t) \rightarrow C^\infty(S^1, V_n)$$

be the Poisson operator corresponding to $\{, \}_i$ at u for $i = 1, 2$, i.e., $(J_i)_u$ is defined by

$$\{F_1, F_2\}_i(u) = \langle (J_i)_u(\nabla F_1(u)), \nabla F_2(u) \rangle.$$

Then the Hamiltonian equation for a functional $H : C^\infty(S^1, V_n) \rightarrow \mathbb{R}$ with respect to $\{, \}_i$ is

$$u_t = (J_i)_u(\nabla H(u)).$$

Next we compute the formula for the Poisson operator J_1 .

Proposition 5.2. *The Poisson operator $(J_1)_u : C^\infty(S^1, V_n^t) \rightarrow C^\infty(S^1, V_n)$ is of the form $(J_1)_u(\xi) = \sum_{i=1}^{n-1} (L_i)_u(\xi) e_{in}$ with*

$$(L_i)_u(\xi) = n(\xi_{n-i})_x + k_i(\xi_1, \dots, \xi_{n-i-1}),$$

where $\xi = \sum_{i=1}^{n-1} \xi_i e_{ni}$ and k_i 's are linear differential operators with differential polynomials of u as coefficients.

Proof. By Theorem 2.3, entries of $P_u(v)$ are differential polynomials of u and v . So we can use integration by parts to compute $(J_1)_u$. We proceed as follows: Let $u = \sum_{i=1}^{n-1} u_i e_{in}$, and

$$\begin{aligned}\xi &= \sum_{i=1}^{n-1} \xi_i e_{ni} := \nabla F_1(u), \quad \eta = \sum_{i=1}^n \eta_i e_{ni} := \nabla F_2(u), \\ C &= (C_{ij}) = P_u(\xi), \quad D = (D_{ij}) = P_u(\eta).\end{aligned}$$

Then we have

$$\{F_1, F_2\}_1(u) = \langle [e_{1n}, C], D \rangle = \oint \sum_{i=1}^n C_{ni} D_{i1} - C_{i1} D_{ni} dx.$$

Let $g : \mathbb{R} \rightarrow SL(n, \mathbb{R})$ be a solution of $g^{-1}g_x = b + u$, γ the first column of g . Then $\gamma \in \mathcal{M}_n(\mathbb{R})$, g is the central affine moving frame along γ , and $u = \Psi(\gamma)$. By Corollary 2.8, there is a $\delta\gamma$ tangent to $\mathcal{M}_n(\mathbb{R})$ at γ such that $C = g^{-1}\delta g$, where $\delta g = (\delta\gamma, \dots, (\delta\gamma)_x^{(n-1)})$. Since $\delta\gamma = \sum_{i=1}^n C_{i1}\gamma_x^{(i-1)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$, it follows from Theorem 2.3 that there exists a differential polynomial ϕ_0 such that

$$C_{11} = \phi_0(C_{21}, \dots, C_{n1}) = f(\xi_1, \dots, \xi_{n-1}).$$

By definition, $C_{ni} = \xi_i$ and $D_{ni} = \eta_i$. By Theorem 2.3, there is a differential polynomial f_n such that

$$C_{nn} = f_n(\xi_1, \dots, \xi_{n-1}), \quad D_{nn} = f_n(\eta_1, \dots, \eta_{n-1}).$$

We then use integration by parts to write down the Poisson operator $(J_1)_u$. To get $(L_j)_u(\xi)$, we only need to calculate the terms involving $\eta_j = D_{nj}$ in $\sum_{i=1}^n C_{i1}D_{ni} - C_{ni}D_{i1}$. We use (2.9) to compute these terms as follows:

$$\begin{aligned} & D_{nj}C_{j1} + C_{n1}(D_{nn} - D_{11}) - \sum_{i=1}^{n+1-j} C_{ni}D_{i1} \\ &= \eta_j C_{j1} + \xi_1 \left(\sum_{i=1}^{n-1} D'_{i+1,i} \right) - C_{n,n+1-j}D_{n+1-j,1} - \sum_{i=1}^{n-j} C_{ni}D_{i1} \\ &= \eta_j(\xi_{n+1-j} - (n-j)\xi'_{n-j} + \phi_{n-j}(u, \xi_1, \dots, \xi_{n-j-1})) + \xi_1 \sum_{i=1}^{n-1} D'_{i+1,i} \\ &\quad - \xi_{n+1-j}(\eta_j - (j-1)\eta'_{j-1} + \phi_{j-1}(u, \eta_1, \dots, \eta_{j-2})) - \sum_{i=1}^{n-j} \xi_i D_{i1}. \end{aligned}$$

Note that $\xi_1 \sum_{i=1}^{n-1} D'_{i+1,i}$ only depends on $\xi_1, \eta_1, \dots, \eta_{n-1}$, and $\xi_i D_{i1}$ is a differential polynomial in u, ξ_i and $\eta_1, \dots, \eta_{n+1-i}$ for each i . Therefore, to consider the term ξ_{n-j} in the coefficient of η_j in $\sum_{i=1}^{n-j} \xi_i D_{i1}$, we only need to calculate $\xi_{n-j}D_{n-j,1}$. Again, use $D_{n-j,1} = \eta_{j+1} - j\eta'_j + \phi_j(u, \eta_1, \dots, \eta_{j-1})$ and integration by parts to see that the coefficients of η_j is $-n\xi'_{n-j}$ plus a differential operator depending on $u, \xi_1, \dots, \xi_{n-j-1}$. \square

Corollary 5.3. *The dimension of the kernel of $(J_1)_u$ is $n-1$.*

Proof. If $(J_1)_u(\xi) = 0$ with $\xi = \sum_{i=1}^{n-1} \xi_i e_{ni}$, then Proposition 5.2 implies that $(L_{n-1})_u(\xi) = n(\xi_1)_x = 0$. So $\xi_1 = c_1$ a constant. Use $(L_{n-2})_u(\xi) = n(\xi_2)_x + k_{n-2}(\xi_1) = 0$ to see that $\xi_2 = -\frac{1}{n}k_{n-2}(c_1)x + c_2$ for some constant c_2 . The corollary follows from induction. \square

Next we compute the formula for the second Poisson operator J_2 .

Proposition 5.4. *The Poisson operator $(J_2)_u : C^\infty(S^1, V_n^t) \rightarrow C^\infty(S^1, V_n)$ is*

$$(J_2)_u(v) = [\partial_x + b + u, P_u(v)]. \quad (5.5)$$

Proof. By definition of P_u , we have $[\partial_x + b + u, P_u(\nabla F_1(u))] \in C^\infty(S^1, V_n)$. So we have

$$\langle [\partial_x + b + u, P_u(\nabla F_1(u))], P_u(\nabla F_2(u)) \rangle = \langle [\partial_x + b + u, P_u(\nabla F_1(u))], \nabla F_2(u) \rangle.$$

Hence the second Poisson structure can be written as

$$\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, P_u(\nabla F_1(u))], \nabla F_2(u) \rangle,$$

which proves that $(J_2)_u(v) = [\partial_x + b + u, P_u(v)]$ is the second Poisson operator. \square

In the following examples, we compute explicit formulas for J_1 and J_2 for small n .

Example 5.5. For $n = 2$, write $u = qe_{21}$, $\nabla F_1(u) = \xi e_{12}$, and $\nabla F_2(u) = \eta e_{21}$, then

$$\begin{aligned} P_u(\nabla F_1(u)) &= \begin{pmatrix} -\frac{1}{2}\xi_x & -\frac{1}{2}\xi_{xx} + q\xi \\ \xi & \frac{1}{2}\xi_x \end{pmatrix}, \\ P_u(\nabla F_2(u)) &= \begin{pmatrix} -\frac{1}{2}\eta_x & -\frac{1}{2}\eta_{xx} + q\eta \\ \eta & \frac{1}{2}\eta_x \end{pmatrix}. \end{aligned}$$

So we get

$$\begin{aligned} \{F_1, F_2\}_1(u) &= 2 \oint \xi' \eta dx, \\ \{F_1, F_2\}_2(u) &= - \oint \left(\frac{1}{2}\xi_x^{(3)} - 2q\xi_x - q_x\xi \right) \eta dx, \end{aligned}$$

and the corresponding Poisson operators are

$$\begin{aligned} (J_1)_u(\xi e_{21}) &= 2\xi_x e_{12}, \\ (J_2)_u(\xi e_{21}) &= \left(-\frac{1}{2}\xi_x^{(3)} + 2q\xi_x + q_x\xi \right) e_{12}. \end{aligned}$$

These are the known Poisson structures for the KdV hierarchy.

Example 5.6. For $n = 3$, write $u = u_1 e_{13} + u_2 e_{23}$, $\xi = \nabla F_1(u) = \xi_1 e_{31} + \xi_2 e_{32}$, $\eta = \nabla F_2(u) = \eta_1 e_{31} + \eta_2 e_{32}$, $C = P_u(\xi) = (C_{ij})$, and $D = P_u(\eta) = (D_{ij})$. Use the algorithm given in the proof of Theorem 2.3 to compute $P_u(v)$ for $v = v_1 e_{31} + v_2 e_{32}$ and get

$$P_u(v) = \begin{pmatrix} -v_2' + \frac{2}{3}v_1'' - \frac{2}{3}u_2 v_1 & p_{12} & p_{13} \\ v_2 - v_1' & p_{22} & p_{33} \\ v_1 & v_2 & v_2' - \frac{1}{3}v_1'' + \frac{1}{3}u_2 v_1 \end{pmatrix}, \quad (5.6)$$

where

$$\begin{aligned} p_{12} &= -v_2'' + \frac{2}{3}v_1^{(3)} - \frac{2}{3}(u_2 v_1)' + u_1 v_1, \\ p_{13} &= -v_2^{(3)} + \frac{2}{3}v_1^{(4)} - \frac{2}{3}(u_2 v_1)'' + (u_1 v_1)' + u_1 v_2, \\ p_{22} &= -\frac{1}{3}v_1'' + \frac{1}{3}u_2 v_1, \\ p_{23} &= -v_2'' + \frac{1}{3}v_1^{(3)} - \frac{1}{3}(u_2 v_1)' + u_2 v_2 + u_1 v_1. \end{aligned}$$

Here we use y' to denote y_x . Then integration by part gives

$$\{F_1, F_2\}_1(u) = \oint \sum_{i=1}^3 C_{3i} D_{i1} - C_{i1} D_{3i} dx = 3 \oint (\xi_1' \eta_2 + \xi_2' \eta_1) dx.$$

Hence

$$(J_1)_u(\xi_1 e_{31} + \xi_2 e_{32}) = 3(\xi_2' e_{13} + \xi_1' e_{23}).$$

This formula was also obtained in [3].

We use (5.5) and a direct computation to see that

$$(J_2)_u(\xi) = (C'_{13} + u_1(C_{33} - C_{11}) - u_2C_{12})e_{13} \\ + (C'_{23} + C_{13} + u_2(C_{33} - C_{22}) - u_1C_{21})e_{23}.$$

This gives

$$(J_2)_u(\xi_1e_{31} + \xi_2e_{32}) = (A_1)_u(\xi)e_{13} + (A_2)_u(\xi)e_{23},$$

where

$$(A_1)_u(\xi) = \frac{2}{3}\xi_1^{(5)} - \xi_2^{(4)} - \frac{2}{3}(u_2\xi_1)^{(3)} - \frac{2}{3}u_2\xi_1^{(3)} + u_1''\xi_1 + 2u_1'\xi_1' \\ + 3u_1\xi_2' + u_1'\xi_2 + u_2\xi_2'' + \frac{2}{3}u_2(u_2\xi_1)'. \\ (A_2)_u(\xi) = \xi_1^{(4)} - 2\xi_2^{(3)} - (u_2\xi_1)'' + 2u_2\xi_2' + u_2'\xi_2 + u_1\xi_1' + 2u_1'\xi_1.$$

Example 5.7. For $n = 4$, let $u = \sum_{i=1}^3 u_i e_{i4}$, $\xi = \nabla F_1(u) = \sum_{i=1}^3 \xi_i e_{4i}$, $\eta = \nabla F_2(u) = \sum_{i=1}^3 \eta_i e_{4i}$, $P_u(\xi) = (C_{ij})$ and $P_u(\eta) = (D_{ij})$. Then the algorithm given in the proof of Theorem 2.3 gives

$$P_u(v) = \begin{pmatrix} a_{11} & * & * & * \\ v_3 - 2v_2' + v_1'' - u_3v_1 & * & * & * \\ v_2 - v_1' & * & * & * \\ v_1 & v_2 & v_3 & a_{44} \end{pmatrix},$$

where $v = \sum_{i=1}^3 v_i e_{4i}$, and

$$a_{11} = -\frac{1}{4}(3v_1^{(3)} - 8v_2'' + 6v_3' - 3(u_3v_1)' + 3u_2v_1 + 2u_3v_2), \\ a_{44} = \frac{1}{4}v_1^{(3)} - v_2'' + \frac{3}{2}v_3' - \frac{1}{4}(u_3v_1)' + \frac{1}{4}u_2v_1 + \frac{1}{2}u_3v_2.$$

So the first Poisson structure is

$$\{F_1, F_2\}_1(u) = \oint \sum_{j=1}^4 (C_{4j}D_{j1} - C_{j1}D_{4j})dx \\ = 4 \oint (\xi_3' - \frac{1}{2}\xi_2'' + \frac{1}{2}\xi_1^{(3)} - \frac{1}{4}(2u_3\xi_1' + u_3'\xi_1))\eta_1 + (\xi_2' + \frac{1}{2}\xi_1'')\eta_2 + \xi_1'\eta_3 dx.$$

Therefore,

$$(J_1)_u(\xi) = \begin{pmatrix} 4\xi_3' - 2\xi_2'' + 2\xi_1''' - 2u_3\xi_1' - u_3'\xi_1 \\ 4\xi_2' + 2\xi_1'' \\ 4\xi_1' \end{pmatrix}.$$

The formula for $(J_2)_u$ can be computed in a similar way as in the case $n = 3$, but it is very long and we do not include here.

Next we review the construction of conservation laws of the $A_{n-1}^{(1)}$ -KdV hierarchy in [5].

Theorem 5.8. ([5]) *Given $u \in C^\infty(\mathbb{R}, V_n)$, then there exists a unique $T = \sum_{i=0}^\infty T_i \lambda^{-i}$ such that $T_0 \in C^\infty(\mathbb{R}, N_n^+)$, the first column of T is e_1 , and*

$$T(\partial_x + J + u)T^{-1} = \partial_x + J + \sum_{i=0}^\infty f_i(u)J^{-i}, \quad f_i(u) \in C^\infty(S^1, \mathbb{R}). \quad (5.7)$$

Moreover, let H_i the functional on $C^\infty(S^1, V_n)$ defined by

$$H_i(u) = n \oint f_i(u) dx. \quad (5.8)$$

Then we have

- (a) $\nabla H_j(u) = \pi_0(Y_{j,0}(u)) = \pi_0(Z_{j,0}(u))$,
- (b) the j -th $A_{n-1}^{(1)}$ -KdV flow (2.5) is the Hamiltonian equation of H_j with respect to $\{\cdot, \cdot\}_2$,
- (c) the j -th $A_{n-1}^{(1)}$ -KdV flow (2.5) is the Hamiltonian equation of H_{n+j} with respect to $\{\cdot, \cdot\}_1$,

where $\pi_0 : sl(n, \mathbb{R}) \rightarrow V_n^t$ is the canonical projection and $Y_{j,0}(u)$ and $Z_{j,0}(u)$ are defined by (2.2) and (2.4) respectively.

Example 5.9. We use (5.7) to compute f_i explicitly. For example, for $n = 3$ and $u = u_1 e_{13} + u_2 e_{23}$, equation (5.7) implies that

$$\begin{aligned} H_1(u) &= \oint u_2 dx, \quad H_2(u) = \oint u_1 dx, \quad H_4(u) = -\frac{1}{3} \oint u_1 u_2 dx, \\ \nabla H_1(u) &= e_{32}, \quad \nabla H_2(u) = e_{31}, \quad \nabla H_4(u) = -\frac{1}{3} u_2 e_{31} - \frac{1}{3} u_1 e_{32}. \end{aligned}$$

Since $P_u(\nabla H(u)) = g^{-1} \delta g$ with $\pi_0(g^{-1} \delta g) = \nabla H(u)$, where π_0 is the canonical projection defined in Definition 2.7, use (5.6) to see that

$$\begin{aligned} P_u(\nabla H_1(u)) &= \begin{pmatrix} 0 & 0 & u_1 \\ 1 & 0 & u_2 \\ 0 & 1 & 0 \end{pmatrix}, \\ Z_{2,0}(u) = P_u(\nabla H_2(u)) &= \begin{pmatrix} -\frac{2}{3}u_2 & u_1 - \frac{2}{3}u_2' & u_1' - \frac{2}{3}u_2'' \\ 0 & \frac{1}{3}u_2 & u_1 - \frac{1}{3}u_2' \\ 1 & 0 & \frac{1}{3}u_2 \end{pmatrix}, \\ Z_{4,0}(u) = P_u(\nabla H_4(u)) &= \begin{pmatrix} \frac{-2u_2^{(2)} + 3u_1' + 2u_2^2}{9} & * & * \\ \frac{u_2' - u_1}{3} & \frac{u_2'' - u_2^2}{9} & 0 \\ -\frac{u_2}{3} & -\frac{u_1}{3} & \frac{u_2'' - 3u_1' - u_2^2}{9} \end{pmatrix}. \end{aligned}$$

For general n , we have

$$\begin{aligned} H_1(u) &= \oint u_{n-1} dx, \\ H_2(u) &= \oint u_{n-2} dx, \\ H_3(u) &= \oint u_{n-3} + \frac{n-3}{2n} u_{n-1}^2 dx. \end{aligned}$$

Next we use a similar computation as in [10] for the $n \times n$ AKNS hierarchy to give another method to construct conservation laws for the $A_{n-1}^{(1)}$ -KdV hierarchy from the solution $Y(u, \lambda)$ of (2.1).

Theorem 5.10. *Given $u \in \mathbb{C}^\infty(S^1 \times \mathbb{R}, V_n)$, let $Y(u, \lambda)$ be the solution of (2.1), $N(u, \lambda) = \lambda^{-1} Y^j(u, \lambda)$, and $\langle \cdot, \cdot \rangle$ the bi-linear form on $C^\infty(S^1, sl(n, \mathbb{R}))$ defined by (5.1). Then we have*

$$\left\langle \frac{\partial N(u, \lambda)}{\partial \lambda}, \delta u \right\rangle = \delta \langle N(u, \lambda), e_{1n} \rangle = \delta \langle \lambda^{-1} Y^j(u, \lambda), e_{1n} \rangle$$

for any variation δu .

Proof. Choose M such that $M(\partial_x + J + u)M^{-1} = \partial_x + J$, i.e.,

$$M^{-1}M_x + M^{-1}JM = J + u.$$

Then we have $Y(u, \lambda) = M^{-1}JM$ and $Y^j(u, \lambda) = M^{-1}J^jM$.

Set $\xi = M^{-1}M_\lambda$ and $\eta = M^{-1}\delta M$. Direct computations give the following formulas:

$$\begin{aligned} \delta Y^j(u, \lambda) &= [Y^j(u, \lambda), \eta], \\ (Y^j)_x(u, \lambda) &= [Y^j(u, \lambda), J + u], \\ \frac{\partial Y^j(u, \lambda)}{\partial \lambda} &= [Y^j(u, \lambda), \xi] + M^{-1}(b^t)^{n-j}M, \\ \delta u &= \eta_x + [J + u, \eta], \\ \xi_x + [J + u, \xi] &= e_{1n} - M^{-1}e_{1n}M. \end{aligned}$$

Use the above formulas to compute to get

$$\begin{aligned} \left\langle \frac{\partial Y^j}{\partial \lambda}, \delta u \right\rangle &= \langle [Y^j, \xi] + M^{-1}(J^j)_\lambda M, \eta_x + [J + u, \eta] \rangle \\ &= -\langle [Y_x^j, \xi], \eta \rangle - \langle [Y^j(u, \lambda), \xi_x], \eta \rangle \\ &\quad - \langle (M^{-1}(J^j)_\lambda M)_x, \eta \rangle + \langle M^{-1}(J^j)_\lambda M, [J + u, \eta] \rangle \\ &= \langle [Y^j, \eta], e_{1n} \rangle + \langle \eta, M^{-1}([(J^j)_\lambda, J] - [e_{1n}, J^j])M \rangle. \end{aligned}$$

Since $[J^j, J] = 0$, we have $[(J^j)_\lambda, J] + [J^j, e_{1n}] = 0$. So we have

$$\left\langle \frac{\partial Y^j(u, \lambda)}{\partial \lambda}, \delta u \right\rangle = \delta \langle Y^j(u, \lambda), e_{1n} \rangle.$$

Therefore

$$\begin{aligned}
\left\langle \frac{\partial N(u, \lambda)}{\partial \lambda}, \delta u \right\rangle &= -\lambda^{-2} \langle Y^j, \delta u \rangle + \lambda^{-1} \left\langle \frac{\partial Y^j}{\partial \lambda}, \delta u \right\rangle \\
&= -\lambda^{-2} \langle Y^j, \eta_x + [J + u, \eta] \rangle + \lambda^{-1} \left\langle \frac{\partial Y^j}{\partial \lambda}, \delta u \right\rangle \\
&= \lambda^{-2} (\langle (Y^j)_x, \eta \rangle - \langle Y^j, [J + u, \eta] \rangle) + \lambda^{-1} \left\langle \frac{\partial Y^j}{\partial \lambda}, \delta u \right\rangle \\
&= \lambda^{-1} \left\langle \frac{\partial Y^j}{\partial \lambda}, \delta u \right\rangle
\end{aligned}$$

This proves the theorem. \square

Since $Y^n(u, \lambda) = M^{-1} J^n M = \lambda I_n$, we have $Y^{nk+j}(u, \lambda) = \lambda^k Y^j(u, \lambda)$. So

$$Y_{nk+j,0}(u) = Y_{j,-k}(u).$$

Recall that if the gradient of a functional at u is $\pi_0(Y_{j,0}(u))$ then it is the Hamiltonian of the j -th $A_{n-1}^{(1)}$ -KdV flow, $u_t = [\partial_x + b + u, P_u(\pi_0(Y_{j,0}(u)))]$. Therefore we have the following.

Corollary 5.11. *Let $F_{j,i} : C^\infty(S^1, V_n) \rightarrow \mathbb{R}$ be the functional defined by*

$$F_{j,i}(u) = -\frac{1}{i+1} \oint \text{tr}(Y_{j,-(i+1)}(u) e_{1n}) dx,$$

where $Y_{j,-(i+1)}(u)$ is the coefficient of $\lambda^{-(i+1)}$ of $Y^j(u, \lambda)$ and $Y(u, \lambda)$ is as in Theorem 5.10. Then $\nabla F_{j,i}(u) = \pi_0(Y_{j,-i}(u)) = \pi_0(Y_{ni+j,0}(u))$ and the Hamiltonian equation for $F_{j,i}$ with respect to $\{, \}_2$ is the $(ni+j)$ -th $A_{n-1}^{(1)}$ -KdV flow.

Next we compute the Casimirs of $\{, \}_1$.

Theorem 5.12. *Let H_j be the functional defined by (5.8) for the $A_{n-1}^{(1)}$ -KdV hierarchy. Then $(J_1)_u(\nabla H_j(u)) = 0, 1 \leq j \leq n-1$. In other words, H_1, \dots, H_{n-1} are Casimirs of $\{, \}_1$. Moreover, $\text{Ker}((J_1)_u)$ is equal to the span of $\{\nabla H_1(u), \dots, \nabla H_{n-1}(u)\}$.*

Proof. Let

$$Y(u, \lambda) = e_{1n} \lambda + Y_{1,0}(u) + Y_{1,1}(u) \lambda^{-1} + \dots$$

be the solution of (2.1) for u , and $Y_{j,0}(u)$ the constant term of $Y(u, \lambda)^j$ as a power series in λ . We claim that $(J_1)_u(\pi_0(Y_{j,0}(u))) = 0$ for $1 \leq j \leq n-1$. It follows from the expansion of $Y(u, \lambda)$ that we have

$$Y(u, \lambda)^j = (b^t)^{n-j} \lambda + Y_{j,0}(u) + Y_{j,1}(u) \lambda^{-1} + \dots, \quad 1 \leq j \leq n-1.$$

Since $[\partial_x + J + u, Y(u, \lambda)^j] = 0$, $[\partial_x + b + u, (b^t)^{n-j}] = [Y_{j,0}(u), e_{1n}]$. Recall that $Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u)$ for some unique $\zeta_j(u) \in C^\infty(S^1, \mathcal{N}_n^+)$. But $[\eta_j(u), e_{1n}] = 0$. So we get

$$[Z_{j,0}(u), e_{1n}] = [Y_{j,0}(u), e_{1n}] = [\partial_x + b + u, (b^t)^{n-j}].$$

By Corollary 2.9, $Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u)))$. Let $\xi_1 = \pi_0(Z_{j,0}(u))$, $\xi_2 \in C^\infty(S^1, V_n^t)$. Then we have

$$\begin{aligned} \langle [Z_{j,0}(u), e_{1n}], P_u(\xi_2) \rangle &= \langle [\partial_x + b + u, (b^t)^{n-j}], P_u(\xi_2) \rangle \\ &= -\langle (b^t)^{n-j}, [\partial_x + b + u, P_u(\xi_2)] \rangle, \end{aligned}$$

which is zero because $[\partial_x + b + u, P_u(\xi_2)] \in V_n$ by definition of P_u and $(b^t)^{n-j}$ is in \mathcal{N}_n^+ . This proves that $\langle (J_1)_u(\pi_0(Z_{j,0}(u))), \xi_2 \rangle = 0$ for all $\xi_2 \in V_n^t$. Hence $\pi_0(Z_{j,0}(u))$ lies in the kernel of $(J_1)_u$. By Theorem 5.8, $\nabla H_j(u) = \pi_0(Z_{j,0}(u))$ for $1 \leq j \leq n-1$. So we get $(J_1)_u(\nabla H_j(u)) = 0, 1 \leq j \leq n-1$. \square

Below we derive some properties of the central affine curvature map Ψ and use them to compute the kernel of $(J_2)_u$.

Proposition 5.13. *Let $\Psi : \mathcal{M}_n(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}, V_n)$ be the central affine curvature map and $u = \Psi(\gamma) = \sum_{i=1}^{n-1} u_i e_{in}$. Then*

$$d\Psi_\gamma(\delta\gamma) = [\partial_x + b + u, g^{-1}\delta g] = (J_2)_u(\pi_0(g^{-1}\delta g)),$$

where $g = (\gamma, \gamma_x, \dots, \gamma_x^{(n-1)})$ is the central affine frame for γ , $\pi_0 : sl(n, \mathbb{R}) \rightarrow V_n^t$ is the canonical projection defined in Definition 2.7, and

$$\delta g = (\delta\gamma, (\delta\gamma)_x, \dots, (\delta\gamma)_x^{(n-1)}).$$

Proof. Take variation of $g^{-1}g_x = b + u$ to get

$$\delta u = -(g^{-1}\delta g)(b + u) + g^{-1}(\delta g)_x.$$

Set $\eta = g^{-1}\delta g$ and compute directly η_x to get $\eta_x = -[b + u, \eta] + \delta u$. But $\Psi(\gamma) = g^{-1}g_x$, where $g = (\gamma, \gamma_x, \dots, \gamma_x^{(n-1)})$. Hence $d\Psi_\gamma(\delta\gamma) = \delta u = [\partial_x + b + u, g^{-1}\delta g]$. \square

Recall that $\Psi(\gamma_1) = \Psi(\gamma_2)$ if and only if there exists $c \in SL(n, \mathbb{R})$ such that $\gamma_2 = c\gamma_1$, where Ψ is the central affine curvature map. So we have

Proposition 5.14. *Let $\Psi : \mathcal{M}_n(S^1) \rightarrow C^\infty(S^1, V_n)$ be the central affine curvature map, $\Psi(\gamma) = u$, and g the central affine moving frame along $\gamma \in \mathcal{M}_n(S^1)$. Then*

- (1) $\text{Ker}(d\Psi_\gamma) = \{c_0\gamma \mid c_0 \in sl(n, \mathbb{C})\}$,
- (2) $\Psi^{-1}(\Psi(\gamma))$ is the $SL(n, \mathbb{R})$ -orbit at γ .

Corollary 5.15. *Let $u \in C^\infty(S^1, V_n)$, and $g : S^1 \rightarrow GL(n, \mathbb{R})$ such that $g^{-1}g_x = b + u$, where $b = \sum_{i=1}^{n-1} e_{i+1,i}$. Let $v \in C^\infty(S^1, V_n^t)$. If $(J_2)_u(v) = 0$, then there is a constant $c_0 \in sl(n, \mathbb{R})$ such that $v = \pi_0(g^{-1}c_0g)$, where $\pi_0 : sl(n, \mathbb{R}) \rightarrow V_n^t$ is the canonical projection.*

Proof. Let γ denote the first column of g . The equation $g_x = g(b + u)$ implies that $g = (\gamma, \gamma_x, \dots, \gamma_x^{(n-1)})$. By Corollary 2.8, there exist $\delta\gamma \in T\mathcal{M}_n(\mathbb{R})_\gamma$ such that $P_u(v) = g^{-1}\delta g$, where $\delta g = (\delta\gamma, \dots, (\delta\gamma)_x^{(n-1)})$. So

$$(J_2)_u(v) = [\partial_x + b + u, g^{-1}\delta g] = d\Psi_\gamma(\delta\gamma) = 0.$$

By Proposition 5.14, there exists content $c_0 \in sl(n, \mathbb{R})$ such that $\delta\gamma = c_0\gamma$. Therefore

$$\delta g = (c_0\gamma, \dots, (c_0\gamma)_x^{(n-1)}) = c_0(\gamma, \dots, \gamma_x^{(n-1)}) = c_0g.$$

This proves that $g^{-1}\delta g = g^{-1}c_0g$. Since $v = \pi_0(P_u(v))$, $v = \pi_0(g^{-1}c_0g)$. \square

It is known that $\{, \}_1$ and $\{, \}_2$ are compatible (cf. [5]), i.e. $c_1\{, \}_1 + c_2\{, \}_2$ is a Poisson structure on $C^\infty(S^1, V_n)$ for all real constants c_1, c_2 . It is standard in the literature (cf. [7], [12]) that we can use these two compatible Poisson structures to generate a sequence of Poisson structures:

$$\{F_1, F_2\}_j(u) = \langle (J_j)_u(\nabla F_1(u)), \nabla F_2(u) \rangle,$$

where

$$J_j = J_2(J_1^{-1}J_2)^{j-2}. \quad (5.9)$$

Moreover, the $(nk + j)$ -th $A_{n-1}^{(1)}$ -KdV flow is

$$u_{t_{nk+j}} = (J_{k+2})_u(\nabla H_j(u)) = J_2(J_1^{-1}J_2)^k(\nabla H_j(u)),$$

where H_j is the functional on $C^\infty(S^1, V_n)$ defined by (5.8).

6. BI-HAMILTONIAN STRUCTURE FOR CENTRAL AFFINE CURVE FLOWS

The pull back $\{, \}_j^\wedge$ of the Poisson structure $\{, \}_j$ to $\mathcal{M}_n(S^1)$ via the central affine curvature map Ψ is for functions of the form $F \circ \Psi$, where F is a functional on $C^\infty(S^1, V_n)$. In other words, $\{, \}_j^\wedge$ is defined by

$$\{F_1 \circ \Psi, F_2 \circ \Psi\}_j^\wedge = \{F_1, F_2\}_j \circ \Psi$$

for functionals F_1, F_2 on $C^\infty(S^1, V_n)$. Let \hat{w}_j be the 2-form defined by $\{, \}_j^\wedge$, i.e.,

$$(\hat{w}_j)_\gamma(X_1(\gamma), X_2(\gamma)) = \{\hat{F}_1, \hat{F}_2\}_j^\wedge(\gamma), \quad (6.1)$$

where $X_i(\gamma)$ is the Hamiltonian vector field for $\hat{F}_i = F_i \circ \Psi$ with respect to $\{, \}_j^\wedge$, $i = 1, 2$. In this section, we

- (1) show that \hat{w}_2 and \hat{w}_3 are the pull backs of symplectic forms on certain co-Adjoint orbits.
- (2) prove that the 2-forms \hat{w}_2 and \hat{w}_3 induce symplectic forms on the orbit spaces $\frac{\mathcal{M}_n(S^1)}{SL(n, \mathbb{R})}$ and $\frac{\mathcal{M}_n(S^1)}{SL(n, \mathbb{R}) \times \mathbb{R}^{n-1}}$ respectively, where $SL(n, \mathbb{R})$ acts on $\mathcal{M}_n(S^1)$ by $c \cdot \gamma = c\gamma$ and the \mathbb{R}^{n-1} -action on $\mathcal{M}_n(S^1)$ is generated by the first $(n-1)$ central affine curve flows (2.11).

Proposition 6.1. *Fix j . Let H be a functional on $C^\infty(S^1, V_n)$, Ψ the central affine curvature map, and δ the Hamiltonian vector field for $\hat{H} = H \circ \Psi$ with respect to $\{, \}_j^\wedge$. Then for $\gamma \in \mathcal{M}_n(S^1)$ we have*

$$[\partial_x + b + u, g^{-1}\delta g] = (J_j)_u(\nabla H(u)),$$

where g and u are the central affine moving frame and central affine curvature along γ respectively and $\delta g = (\delta\gamma, \dots, (\delta\gamma)_x^{(n-1)})$.

Proof. Since $\{, \}_j^\wedge$ is the pull back of $\{, \}_j$, we have

$$d\Psi_\gamma(\delta\gamma) = (J_j)_u(\nabla H(u)).$$

By Proposition 5.13, $d\Psi_\gamma(\delta\gamma) = [\partial_x + b + u, g^{-1}\delta g]$. \square

Corollary 6.2. *Let H_j be functionals on $C^\infty(S^1, V_n)$ defined by (5.8), and $\hat{H}_j = H_j \circ \Psi$. Then the j -th central affine curve flow (2.11) is the Hamiltonian equation for \hat{H}_j and \hat{H}_{n+j} with respect to $\{, \}_2^\wedge$ and $\{, \}_1^\wedge$ respectively.*

Corollary 6.3. *Let $1 \leq j \leq n-1$ and $k \geq 0$. Then the $(nk+j)$ -th central affine curve flow on $\mathcal{M}_n(S^1)$ is*

$$\gamma_{t_{nk+j}} = gZ_{nk+j,0}(u)e_1 = g(P_u((J_1^{-1}J_2)_u^k(\nabla H_j(u))))e_1,$$

where g and u are the central affine moving frame and curvature of γ respectively.

Proposition 6.4. *Let F_i be functionals on $C^\infty(S^1, V_n)$, and δ_i the Hamiltonian vector field for $\hat{F}_i = F_i \circ \Psi$ with respect to $\{, \}_j^\wedge$ for $i = 1, 2$. Then*

$$\{\hat{F}_1, \hat{F}_2\}_j^\wedge(\gamma) = -\langle g^{-1}\delta_1 g, (J_2 J_j^{-1} J_2)_u(\pi_0(g^{-1}\delta_2 g)) \rangle,$$

where g is the central affine moving frame along γ , $u = \Psi(\gamma)$, and $\delta_i g = (\delta_i \gamma, (\delta_i \gamma)_x, \dots, (\delta_i \gamma)_x^{(n-1)})$.

Proof. By Proposition 5.13, $d\Psi_\gamma(\delta\gamma) = (J_2)_u(\pi_0(g^{-1}\delta g))$. Since $d\Psi(\delta_i \gamma) = (J_j)_u(\nabla F_i(u))$ for $i = 1, 2$, we get $\nabla F_2(u) = (J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_2 g))$. So we have $(J_j)_u(\nabla F_2(u)) = J_2(\pi_0(g^{-1}\delta_2 g))$. Compute directly to get

$$\begin{aligned} \{\hat{F}_1, \hat{F}_2\}_j^\wedge(\gamma) &= \{F_1, F_2\}_j(u) = \langle (J_j)_u(\nabla F_1(u)), \nabla F_2(u) \rangle \\ &= \langle J_2(\pi_0(g^{-1}\delta_1 g)), (J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_2 g)) \rangle \\ &= -\langle \pi_0(g^{-1}\delta_1 g), (J_2 J_j^{-1} J_2)_u(\pi_0(g^{-1}\delta_2 g)) \rangle \\ &= -\langle g^{-1}\delta_1 g, (J_2 J_j^{-1} J_2)_u(\pi_0(g^{-1}\delta_2 g)) \rangle. \end{aligned}$$

\square

Proposition 6.5. *Let \hat{w}_j be the 2-form on $\mathcal{M}_n(S^1)$ defined by (6.1), i.e.,*

$$(\hat{w}_j)_\gamma(\delta_1 \gamma, \delta_2 \gamma) = -\langle g^{-1}\delta_1 g, (J_2 J_j^{-1} J_2)_u(\pi_0(g^{-1}\delta_2 g)) \rangle,$$

where $\delta_i g = (\delta_i \gamma, \dots, (\delta_i \gamma)_x^{(n-1)})$ for $i = 1, 2$, and g is the central affine moving frame along γ . Then

$$(\hat{w}_2)_\gamma(\delta_1 \gamma, \delta_2 \gamma) = \langle [\partial_x + b + u, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle, \quad (6.2)$$

$$(\hat{w}_3)_\gamma(\delta_1 \gamma, \delta_2 \gamma) = \langle [e_{1n}, g^{-1}\delta_1 g], g^{-1}\delta_2 g \rangle. \quad (6.3)$$

Next we write down $\hat{w}_2(X, Y)$ and $\hat{w}_3(X, Y)$ in terms of determinants involving $X, Y \in T\mathcal{M}_n(S^1)$ and derivatives of X and Y :

Theorem 6.6. *Let X, Y be tangent vectors of $\mathcal{M}_n(S^1)$ at γ . Then*

$$\begin{aligned} (\hat{w}_2)_\gamma(X, Y) &= - \sum_{i=1}^{n-1} \oint \det(\gamma, \dots, \gamma_x^{i-2}, X_x^{(n)}, \gamma_x^{(i)}, \dots, Y_x^{(i-1)}) dx \\ &\quad + \sum_{i,j=1}^{n-1} \oint u_j \det(\gamma, \dots, \gamma_x^{(i-1)}, X_x^{(j-1)}, \gamma_x^{(i)}, \dots, Y_x^{(i-1)}) dx, \\ (\hat{w}_3)_\gamma(X, Y) &= - \sum_{i=1}^{n-1} \oint \det(\gamma, \dots, \gamma_x^{(i-2)}, X, \gamma_x^{(i)}, \dots, Y_x^{(i-1)}) dx \\ &\quad - \sum_{i=1}^{n-1} \oint \det(\gamma, \dots, \gamma_x^{(i-2)}, X_x^{(i-1)}, \gamma_x^{(i)}, \dots, Y) dx. \end{aligned}$$

Proof. Let $\gamma, g, u, \delta_i \gamma, \delta_i g$ be as in Proposition 6.4,

$$C = (C_{ij}) = g^{-1} \delta_1 g, \quad D = (D_{ij}) = g^{-1} \delta_2 g,$$

and C_i, D_i the i -th column of C and D respectively. By Theorem 2.3, we can express C_i 's as differential polynomials in C_1 . Similarly, D_i 's can be expressed as differential polynomials in D_1 . Moreover, $C_i = (C_{1i}, \dots, C_{ni})$ is the coordinate of $(\delta_1 \gamma)_x^{(i-1)}$ with respect to the frame $g = (\gamma, \dots, \gamma_x^{(n-1)})$, i.e., $(\delta_1 \gamma)_x^{(i-1)} = \sum_{k=1}^n C_{ki} \gamma_x^{(k-1)}$.

Recall that if $Y = \sum_{i=1}^n y_i \gamma_x^{(i-1)}$, then $Y' = \sum_{i=1}^n (y'_i + y_{i-1} + u_i y_n) \gamma_x^{(i-1)}$. Write $(\delta_1 \gamma)_x^{(n)} = \sum_{i=1}^n \xi_i (\delta_1 \gamma)_x^{(i-1)}$, Then $\xi = (\xi_1, \dots, \xi_n)^t = C'_n + (b+u)C_n$. By Proposition 6.5, we have

$$\begin{aligned} (\hat{w}_3)_\gamma(\delta_1 \gamma, \delta_2 \gamma) &= \oint \sum_{i=1}^n C_{ni} D_{i1} - C_{i1} D_{ni} dx, \\ (\hat{w}_2)_\gamma(\delta_1 \gamma, \delta_2 \gamma) &= \oint \sum_{i=1}^n (C(b+u))_{in} D_{ni} - (C_x + (b+u)C)_{in} D_{ni} dx \\ &= \oint \sum_{i=1}^n \sum_{j=1}^{n-1} C_{ij} u_j D_{ni} - \sum_{i=1}^n \xi_i D_{ni} dx. \end{aligned}$$

We compute $(\hat{w}_3)_\gamma$ as follows: Let $X = \delta_1 \gamma$, $Y = \delta_2 \gamma$, then

$$\begin{aligned} (\hat{w}_3)_\gamma(X, Y) &= \oint \sum_{i=1}^n C_{ni} D_{i1} - C_{i1} D_{ni} dx \\ &= \oint C_{nn} D_{n1} - C_{n1} D_{nn} + \sum_{i=1}^{n-1} C_{ni} D_{i1} - C_{i1} D_{ni} dx \\ &= \oint C_{n1} \sum_{i=1}^{n-1} D_{ii} - \left(\sum_{i=1}^{n-1} C_{ii} \right) D_{n1} + \sum_{i=1}^{n-1} C_{ni} D_{i1} - C_{i1} D_{ni} dx. \end{aligned}$$

Note that $\det(\gamma, \dots, \gamma_x^{(n-1)}) = 1$ and the k -th column of C and D are the coefficients of $X_x^{(k-1)}$ and $Y_x^{(k-1)}$ written as a linear combination of $\gamma, \dots, \gamma_x^{(n-1)}$. So we have

$$\det(\gamma, \gamma_x, \dots, \gamma_x^{(i-2)}, X_x^{(k-1)}, \gamma_x^{(i)}, \dots, Y_x^{(\ell-1)}) = C_{ik}D_{n\ell} - C_{nk}D_{i\ell}. \quad (6.4)$$

Substitute (6.4) into the above formula for $w_\gamma(X, Y)$ to get the formula for $(\hat{w}_3)_\gamma$ as stated in the theorem.

Use $\text{tr}(C) = \text{tr}(D) = 0$ and (6.4) to get the formula for \hat{w}_2 . \square

Example 6.7. For $n = 2$, Theorem 6.6 gives

$$\begin{aligned} (\hat{w}_2)_\gamma(X, Y) &= - \oint \det(X', Y') + u_1 \det(X, Y) dx, \\ (\hat{w}_3)_\gamma(X, Y) &= -2 \oint \det(X, Y) dx. \end{aligned}$$

These are the 2 forms given in [6] and [8] respectively.

Example 6.8. For $n = 3$, we get

$$(\hat{w}_3)_\gamma(X, Y) = -3 \oint \det(X, \gamma', Y) dx,$$

which is the 2 form given in [3]. We also have

$$\begin{aligned} (\hat{w}_2)_\gamma(X, Y) &= - \oint \det(X''', \gamma', Y) + \det(\gamma, X''', Y') dx \\ &\quad + \oint u_1 (\det(X, \gamma', Y) + \det(\gamma, X, Y')) dx \\ &\quad + \oint u_2 (\det(X', \gamma', Y) + \det(\gamma, X', Y')) dx. \end{aligned}$$

Example 6.9. For $n = 4$, let $|v_1, v_2, v_3, v_4| = \det(v_1, v_2, v_3, v_4)$. Then we have

$$\begin{aligned} (\hat{w}_3)_\gamma(X, Y) &= -2 \oint |X, \gamma', \gamma'', Y| + |\gamma, \gamma', X', Y'| dx, \\ (\hat{w}_2)_\gamma(X, Y) &= - \oint |X^{(4)}, \gamma', \gamma'', Y| + |\gamma, X^{(4)}, \gamma'', Y'| + |\gamma, \gamma', X^{(4)}, Y''| dx \\ &\quad + \oint u_3 (|X'', \gamma', \gamma'', Y| + |\gamma, X'', \gamma'', Y'| + |\gamma, \gamma', X'', Y''|) dx \\ &\quad + \oint u_2 (|X', \gamma', \gamma'', Y| + |\gamma, X', \gamma'', Y'| + |\gamma, \gamma', X', Y''|) dx \\ &\quad + \oint u_1 (|X, \gamma', \gamma'', Y| + |\gamma, X, \gamma'', Y'| + |\gamma, \gamma', X, Y''|) dx. \end{aligned}$$

Next we prove that \hat{w}_2 and \hat{w}_3 are the pull backs of certain co-Adjoint orbit symplectic forms. Let M be the co-Adjoint orbit of G on the dual \mathcal{G}^*

of the Lie algebra \mathcal{G} at $\ell_0 \in \mathcal{G}^*$. The orbit symplectic form on M is defined by

$$\tau_\ell(\tilde{\xi}(\ell), \tilde{\eta}(\ell)) = \ell([\xi, \eta]),$$

where $\ell \in M$, and $\tilde{\xi}$ and $\tilde{\eta}$ are infinitesimal vector fields corresponding to the co-Adjoint action generated by $\xi, \eta \in \mathcal{G}$.

We identify the Adjoint orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ on $C^\infty(\mathbb{R}, sl(n, \mathbb{R}))$ as the co-Adjoint orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ via the non-degenerate bi-linear form (5.1), i.e.,

$$\langle \xi, \eta \rangle = \oint \text{tr}(\xi(x)\eta(x))dx.$$

Theorem 6.10. *Let \mathcal{O}_1 denote the Adjoint orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ on $C^\infty(S^1, sl(n, \mathbb{R}))$ at the constant loop e_{1n} , τ_1 the orbit symplectic form on \mathcal{O}_1 , and \mathfrak{k}_1 the map from $\mathcal{M}_n(S^1)$ to \mathcal{O}_1 defined by $\mathfrak{k}_1(\gamma) = ge_{1n}g^{-1}$, where g is the central affine moving frame along γ . Let \hat{w}_3 be as in (6.3). Then $\mathfrak{k}_1^*\tau_1 = \hat{w}_3$.*

Proof. Given $\xi \in C^\infty(S^1, sl(n, \mathbb{R}))$, a direct computation implies that the infinitesimal vector field is

$$\tilde{\xi}(ge_{1n}g^{-1}) = [\xi, ge_{1n}g^{-1}].$$

So the orbit symplectic structure is

$$(\tau_1)_{ge_{1n}g^{-1}}([\xi, ge_{1n}g^{-1}], [\eta, ge_{1n}g^{-1}]) = \langle ge_{1n}g^{-1}, [\xi, \eta] \rangle.$$

The differential of \mathfrak{k}_1 at γ is

$$d(\mathfrak{k}_1)_\gamma(\delta\gamma) = [\delta g g^{-1}, ge_{1n}g^{-1}].$$

Hence

$$\begin{aligned} (\mathfrak{k}_1^*\tau_1)_\gamma(\delta_1\gamma, \delta_2\gamma) &= (\tau_1)_{ge_{1n}g^{-1}}([\delta_1 g g^{-1}, ge_{1n}g^{-1}], [\delta_2 g g^{-1}, ge_{1n}g^{-1}]) \\ &= \langle ge_{1n}g^{-1}, [(\delta_1 g g^{-1}, (\delta_2 g g^{-1}))] \rangle = \langle e_{1n}, [g^{-1}\delta_1 g, g^{-1}\delta_2 g] \rangle, \end{aligned}$$

which is equal to $(\hat{w}_3)_\gamma(\delta_1\gamma, \delta_2\gamma)$. \square

Let $\mathbb{R}\partial_x + C^\infty(S^1, sl(n, \mathbb{R}))$ denote the Lie algebra with bracket defined by

$$[r_1\partial_x + u, r_2\partial_x + v] = r_1v_x - r_2u_x + [u, v], \quad r_1, r_2 \in \mathbb{R}.$$

It is known (cf. [9], [10]) that the dual of the central extension of the loop algebra $C^\infty(S^1, sl(n, \mathbb{R}))$ defined by the 2-cocycle

$$\rho(\xi, \eta) = \oint \text{tr}(\xi_x(x)\eta(x))dx$$

can be identified as the Lie algebra $\mathbb{R}\partial_x + C^\infty(S^1, sl(n, \mathbb{R}))$. The co-Adjoint action corresponds to the gauge action,

$$g \cdot (\partial_x + u) = g(\partial_x + u)g^{-1} = \partial_x + gug^{-1} - g_xg^{-1}.$$

Theorem 6.11. *Let \mathcal{O}_2 denote the gauge orbit of $C^\infty(S^1, SL(n, \mathbb{R}))$ at ∂_x , τ_2 the orbit symplectic form on \mathcal{O}_2 , and $\mathfrak{k}_2 : \mathcal{M}_n(S^1) \rightarrow \mathcal{O}_2$ the map defined by $\mathfrak{k}_2(\gamma) = g^{-1}g_x$, where g is the central affine moving frame along γ . Let \hat{w}_2 be as in (6.2). Then $\mathfrak{k}_2^*(\tau_2) = \hat{w}_2$.*

Proof. The infinitesimal vector field on \mathcal{O}_2 given by the gauge action for $\xi \in C^\infty(S^1, sl(n, \mathbb{R}))$ is $\tilde{\xi}(\partial_x + v) = -[\partial_x + v, \xi]$. Note that

$$\mathfrak{k}_2(\gamma) = \partial_x + g^{-1}g_x = \partial_x + b + u = \partial_x + b + \Psi(\gamma).$$

By Proposition 5.13,

$$d(\mathfrak{k}_2)_\gamma(\delta\gamma) = d\Psi_\gamma(\delta\gamma) = [\partial_x + b + u, g^{-1}\delta g].$$

Then

$$\begin{aligned} (\mathfrak{k}_2^*\tau_2)_\gamma(\delta_1\gamma, \delta_2\gamma) &= (\tau_2)_{\partial_x+b+u}(d\mathfrak{k}_2(\delta_1\gamma), d\mathfrak{k}_2(\delta_2\gamma)) \\ &= (\tau_2)_{\partial_x+b+u}([\partial_x + b + u, g^{-1}\delta_1g], [\partial_x + b + u, g^{-1}\delta_2g]) \\ &= \langle [\partial_x + b + u, g^{-1}\delta_1g], g^{-1}\delta_2g \rangle, \end{aligned}$$

which is equal to $(\hat{w}_2)_\gamma(\delta_1\gamma, \delta_2\gamma)$. \square

Recall that a *weak symplectic form* on M is a closed 2-form on M such that $w_x(v_1, v_2) = 0$ for all $v_2 \in TM_x$ implies that $v_1 = 0$. If M is of finite dimension, then a weak symplectic form is symplectic. When M is of infinite dimension, a weak symplectic form need not to be non-degenerated, but we can still have the Hamiltonian theory (cf. [1]). Below we show that \hat{w}_2 and \hat{w}_3 induce weak symplectic forms on the orbit spaces $\mathcal{M}_n(S^1)/SL(n, \mathbb{R})$ and $\mathcal{M}_n(S^1)/(SL(n, \mathbb{R}) \times \mathbb{R}^{n-1})$ respectively.

Theorem 6.12. *The 2-form \hat{w}_2 induces a weak symplectic form on the orbit space $\mathcal{M}_n(S^1)/SL(n, \mathbb{R})$.*

Proof. By Theorem 6.11, \hat{w}_2 is a closed 2-form. It follows from (6.2) that $(\hat{w}_2)_\gamma(\delta_1\gamma, \delta_2\gamma) = 0$ for all $\delta_2\gamma$ if and only if $[\partial_x + b + u, g^{-1}\delta_1g] = 0$, where g is the central affine moving frame along γ and $u = \Psi(\gamma)$. By Proposition 5.13,

$$d\Psi_\gamma(\delta_1\gamma) = [\partial_x + b + u, g^{-1}\delta_1g] = (J_2)_u(\pi_0(g^{-1}\delta_1g)) = 0.$$

The theorem follows from Proposition 5.14. \square

We consider the \mathbb{R}^{n-1} -action on $\mathcal{M}_n(S^1)$ generated by the first $(n-1)$ central affine curve flow (2.11). Since the central affine curve flows commute with the $SL(n, \mathbb{R})$ -action, the product group $SL(n, \mathbb{R}) \times \mathbb{R}^{n-1}$ acts on $\mathcal{M}_n(S^1)$.

Theorem 6.13. *The 2-form \hat{w}_3 induces a weak symplectic form on the space $\mathcal{M}_n(S^1)/(SL(n, \mathbb{R}) \times \mathbb{R}^{n-1})$.*

Proof. By Theorem 6.10, \hat{w}_3 is a closed 2-form. The formula of \hat{w}_3 implies that $(\hat{w}_3)_\gamma(\delta_1\gamma, \delta_2\gamma) = 0$ for all $\delta_2\gamma$ in $T\mathcal{M}(S^1)_\gamma$ if and only if

$$(J_1)_u(\pi_0(g^{-1}\delta_1g)) = 0.$$

So $\pi_0(g^{-1}\delta_1 g)$ lies in the kernel of $(J_1)_u$. It follows from Theorem 5.12 that ξ lies in the span of $\nabla H_1(u), \dots, \nabla H_{n-1}(u)$, where H_i 's are the Hamiltonians defined by (5.8).

Let X_1, \dots, X_{n-1} denote the vector fields that generated the first $(n-1)$ central affine curve flows. Then $d\Psi_\gamma(X_i) = (J_2)_u(\nabla H_i(u))$. If $\pi_0(g^{-1}\delta_1 g) = \nabla H_i(u)$ for some $1 \leq i \leq n-1$, then $d\Psi(\delta_1 \gamma) = (J_2)_u(\nabla H_i(u))$. So $\delta_1 \gamma \in X_i(\gamma) + \text{Ker}(d\Psi_\gamma)$. By Proposition 5.14, $\text{Ker}(d\Psi_\gamma) =$ the tangent space of the $SL(n, \mathbb{R})$ -orbit at γ . This finishes the proof. \square

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